

Anti De Sitter Deformation of Quaternionic Analysis and the Second Order Pole

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Abstract

This is a continuation of a series of papers [FL1, FL2, FL3], where we develop quaternionic analysis from the point of view of representation theory of the conformal Lie group and its Lie algebra. In this paper we continue to study the quaternionic analogues of Cauchy's formula for the second order pole. These quaternionic analogues are closely related to regularization of infinities of vacuum polarization diagrams in four-dimensional quantum field theory. In order to add some flexibility, especially when dealing with Cauchy's formula for the second order pole, we introduce a one-parameter deformation of quaternionic analysis. This deformation of quaternions preserves conformal invariance and has a geometric realization as anti de Sitter space sitting inside the five-dimensional Euclidean space. We show that many results of quaternionic analysis – including the Cauchy-Fueter formula – admit a simple and canonical deformation. We conclude this paper with a deformation of the quaternionic analogues of Cauchy's formula for the second order pole.

1 Introduction

Let \mathbb{H} denote the algebra of quaternions

$$\mathbb{H} = 1\mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$$

with the norm

$$N(X) = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2, \quad X = x^0 + ix^1 + jx^2 + kx^3 \in \mathbb{H}.$$

Since the early days of quaternionic analysis, when the quaternionic analogue of complex holomorphic functions was introduced, there was a fundamental question about the natural quaternionic analogue of the ring structure of holomorphic functions¹. In particular, one can ask what is the quaternionic version of the ring of polynomials $\mathbb{C}[z]$ and Laurent polynomials $\mathbb{C}[z, z^{-1}]$. The representation theoretic approach that we have developed in [FL1, FL2, FL3] suggests the most naive candidates for an answer: \mathbb{H} -valued polynomial functions on \mathbb{H} and $\mathbb{H}^\times = \{X \in \mathbb{H}; X \neq 0\}$ respectively:

$$\mathbb{H}[x^0, x^1, x^2, x^3] \quad \text{and} \quad \mathbb{H}[x^0, x^1, x^2, x^3, N(X)^{-1}]. \quad (1)$$

Another option is just to consider \mathbb{R} -valued polynomial functions on \mathbb{H} and \mathbb{H}^\times :

$$\mathbb{R}[x^0, x^1, x^2, x^3] \quad \text{and} \quad \mathbb{R}[x^0, x^1, x^2, x^3, N(X)^{-1}]. \quad (2)$$

¹Some readers may point out the Cauchy-Kovalevskaya product of quaternionic regular functions. But this operation is not satisfactory, since it does not have good invariance properties with respect to the conformal group action.

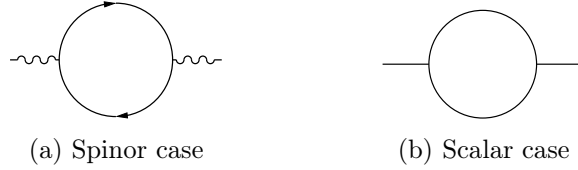


Figure 1: Vacuum polarization diagrams

Clearly, all four of these spaces of functions have natural ring structures. However, these functions are typically neither regular nor harmonic, so all the regular quaternionic structure is lost. Representation theory asserts that all four quaternionic rings yield the so-called middle series of representations of the conformal Lie algebra $\mathfrak{sl}(2, \mathbb{H}) \simeq \mathfrak{so}(5, 1)$ and that there are intertwining maps from tensor products of left regular and right regular polynomials into the two rings (1) and from the tensor products of harmonic polynomials into the two rings (2). It is also natural to complexify the quaternions

$$\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = 1\mathbb{C} \oplus i\mathbb{C} \oplus j\mathbb{C} \oplus k\mathbb{C}$$

and the spaces of polynomial functions on them (1) and (2). This brings us to the main objects of our study:

$$\mathcal{W}^+ = \mathbb{H}_{\mathbb{C}}[z^0, z^1, z^2, z^3], \quad (3)$$

$$\mathcal{W} = \mathbb{H}_{\mathbb{C}}[z^0, z^1, z^2, z^3, N(Z)^{-1}], \quad (4)$$

$$\mathcal{K}^+ = \mathbb{C}[z^0, z^1, z^2, z^3], \quad (5)$$

$$\mathcal{K} = \mathbb{C}[z^0, z^1, z^2, z^3, N(Z)^{-1}] \quad (6)$$

– the two versions of the rings of ordinary and Laurent polynomials in one complex variable.

The relation between the quaternionic ring (5) and harmonic functions yields a reproducing integral formula for functions in \mathcal{K}^+ . On the other hand, the relation between the ring (3) and the regular functions is similar, but instead of a reproducing formula we get an integral expression for a certain second order differential operator

$$\text{Mx} : \mathcal{W}^+ \rightarrow \mathcal{W}^+, \quad \text{Mx} f = \nabla f \nabla - \square f^+.$$

(The operator Mx is directly related to the solutions of the Maxwell equations for the gauge potential.) All of these formulas can be regarded as quaternionic analogues of the Cauchy’s formula for the second order pole

$$f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z) dz}{(z - z_0)^2}$$

(see [FL1] for details). The corresponding formulas for \mathcal{W} and \mathcal{K} are more involved and are the main subject of this paper. They require certain regularizations of infinities which are well known in four-dimensional quantum field theory as “vacuum polarizations” in spinor and scalar cases respectively. They are usually encoded by the Feynman diagrams shown in Figure 1 and play a key role in renormalization theory (see, for example, [Sm]).

In order to explain our reproducing formula in the scalar case, we recall the space \mathcal{H} of harmonic functions on \mathbb{H}^{\times} . It decomposes into two irreducible components:

$$\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+ \quad (7)$$

with respect to the action of the conformal algebra $\mathfrak{sl}(2, \mathbb{H})$. Similarly, we can decompose \mathcal{K} into three irreducible components:

$$\mathcal{K} = \mathcal{K}^- \oplus \mathcal{K}^0 \oplus \mathcal{K}^+. \quad (8)$$

The spaces \mathcal{K}^+ and \mathcal{K}^- have already appeared in [FL1], but the appearance of \mathcal{K}^0 in quaternionic analysis is new. We study equivariant embeddings of \mathcal{K}^- , \mathcal{K}^0 and \mathcal{K}^+ into tensor products $\mathcal{H}^\pm \otimes \mathcal{H}^\pm$. The cases \mathcal{K}^- and \mathcal{K}^+ are fairly straightforward, but the case of \mathcal{K}^0 – which is the core of the scalar vacuum polarization – is more subtle. As a consequence of these equivariant embeddings, we obtain projectors of \mathcal{K} onto its irreducible components. Using these projectors we get a reproducing formula for all functions in \mathcal{K} , which may be loosely stated as follows. Let

$$(I_1 f)(Z_1, Z_2) = \frac{i}{2\pi^3} \int_{W \in U(2)} \frac{f(W) dV}{N(W - Z_1) \cdot N(W - Z_2)}, \quad f \in \mathcal{K}, \quad Z_1, Z_2 \in \mathbb{D}^+ \sqcup \mathbb{D}^-,$$

where \mathbb{D}^+ and \mathbb{D}^- are two certain open regions in $\mathbb{H}_\mathbb{C}$ both having $U(2)$ as their Shilov boundary. Then

$$\begin{aligned} f(Z) = & \lim_{\substack{Z_1, Z_2 \rightarrow Z \\ Z_1 \in \mathbb{D}^+, Z_2 \in \mathbb{D}^+}} (I_1 f)(Z_1, Z_2) - \lim_{\substack{Z_1, Z_2 \rightarrow Z \\ Z_1 \in \mathbb{D}^+, Z_2 \in \mathbb{D}^-}} (I_1 f)(Z_1, Z_2) \\ & - \lim_{\substack{Z_1, Z_2 \rightarrow Z \\ Z_1 \in \mathbb{D}^-, Z_2 \in \mathbb{D}^+}} (I_1 f)(Z_1, Z_2) + \lim_{\substack{Z_1, Z_2 \rightarrow Z \\ Z_1 \in \mathbb{D}^-, Z_2 \in \mathbb{D}^-}} (I_1 f)(Z_1, Z_2), \quad f \in \mathcal{K}, \quad Z \in U(2). \end{aligned} \quad (9)$$

(see Remark 16). A similar formula can be deduced for the operator Mx acting on \mathcal{W} .

The treatment of the projector onto \mathcal{K}^0 and the resulting reproducing formula are not completely satisfactory, since the points Z_1 and Z_2 belong to the non-intersecting domains \mathbb{D}^+ and \mathbb{D}^- . This phenomenon is well known in physics, where it results in the divergence of the Feynman integral corresponding to the scalar vacuum polarization diagram. Physicists have several methods to achieve this isolation of singularity involving introduction of an auxiliary parameter. Depending on the method, this auxiliary parameter can be interpreted as dimension or mass. The former method is incompatible with representation theoretic approach and the latter is better from our point of view, but still breaks the conformal symmetry down to the famous Poincare group. There is, however, a third way to introduce an auxiliary parameter while fully preserving the conformal invariance – namely via anti de Sitter deformation of the flat Minkowski space. This is the method we pursue in the second part of the paper to develop a deformation of quaternionic analysis. First of all, we define a one-parameter family of conformal Laplacians

$$\widetilde{\square}_\mu = \square + \mu^2 (\widetilde{\deg}^2 + \widetilde{\deg}) \quad (10)$$

depending on a real parameter μ , where $\widetilde{\deg}$ denotes the degree operator plus identity and \square is the ordinary Laplacian on \mathbb{H} . As usual, the deformed Laplacian admits a quaternionic factorization into two first order differential operators

$$\widetilde{\square}_\mu = \overrightarrow{\nabla}_\mu (\overrightarrow{\nabla}_\mu - \mu) = \overleftarrow{\nabla}_\mu (\overleftarrow{\nabla}_\mu + \mu), \quad (11)$$

where the arrows indicate that the operator $\overrightarrow{\nabla}_\mu$ is applied to functions on the left and $\overleftarrow{\nabla}_\mu$ is applied on the right. This factorization allows us to define a one-parameter family of left and right regular functions by the requirement

$$\overrightarrow{\nabla}_\mu f = 0 \quad \text{and} \quad g \overleftarrow{\nabla}_\mu = 0.$$

Then we prove analogues of Cauchy-Fueter and Poisson formulas as well as generalize certain other constructions and results from quaternionic analysis. In particular, the Poisson kernel $N(X - Y)^{-1}$ is replaced by the following family of kernels depending on μ

$$\frac{1}{\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4}}, \quad (12)$$

where

$$\hat{X} = (\sqrt{\mu^{-2} + N(X)}, x^0, x^1, x^2, x^3), \quad \hat{Y} = (\sqrt{\mu^{-2} + N(Y)}, y^0, y^1, y^2, y^3)$$

and the 5-dimensional space $\mathbb{R}^{1,4}$ is equipped with an indefinite inner product

$$\langle W, W' \rangle_{1,4} = w^0 w'^0 - w^1 w'^1 - w^2 w'^2 - w^3 w'^3 - w^4 w'^4,$$

$W = (w^0, w^1, w^2, w^3, w^4)$, $W' = (w'^0, w'^1, w'^2, w'^3, w'^4) \in \mathbb{R}^{1,4}$. After developing basics of the anti de Sitter deformation of quaternionic analysis we turn to the treatment of the second order pole.

As the expression for the reproducing kernel (12) indicates, various one-parameter generalizations of results from quaternionic analysis admit a natural geometric interpretation when we identify the space of quaternions with a single sheet of a two-sheeted hyperboloid in $\mathbb{R}^{1,4}$. This hyperboloid is known to physicists as the anti de Sitter space. Thus the anti de Sitter space-time geometry – which has been extensively studied by physicists (see, for example, [BGMT] and references therein) – naturally provides a one-parameter deformation of (classical) quaternionic analysis. We obtain, in particular, a deformation of the representations \mathcal{H}_μ^\pm , \mathcal{K}_μ^\pm and \mathcal{K}_μ^0 of the conformal Lie algebra and find projectors onto these spaces. This brings us back to our original motivation of the one-parameter deformation of quaternionic analysis – finding a representation-theoretic interpretation of the regularization in quantum field theory. This question will be addressed in a subsequent work.

The paper consists of two parts related by a common motivation of development of quaternionic analysis using representation-theoretic methods. In Sections 2-8 we study structures related to the second order pole and in Sections 9-17 we develop the one-parameter deformation of quaternionic analysis using geometry of the anti de Sitter space. In Section 2 we summarize the results of quaternionic analysis that are used in this article and, in particular, introduce the representation (ρ_1, \mathcal{K}) of the conformal algebra $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \simeq \mathfrak{gl}(4, \mathbb{C})$ which is one of the main subject of this work. In Section 3 we give explicit K -types of (ρ_1, \mathcal{K}) , and in Section 4 we show that the representation (ρ_1, \mathcal{K}) decomposes into three irreducible components (8) (Theorem 7). We also prove that the subspaces \mathcal{K}^- , \mathcal{K}^0 and \mathcal{K}^+ are the images under the natural multiplication maps of, respectively, $\mathcal{H}^- \otimes \mathcal{H}^-$, $\mathcal{H}^- \otimes \mathcal{H}^+$ and $\mathcal{H}^+ \otimes \mathcal{H}^+$ (Lemma 8). In Section 5 we make a formal calculation of the reproducing kernel for \mathcal{K}^0 . (Note that the reproducing kernels for \mathcal{K}^+ and \mathcal{K}^- were computed in [FL1].) In Section 6 we study conformally invariant embeddings of the irreducible components \mathcal{K}^\pm and \mathcal{K}^0 into tensor products $\mathcal{H}^\pm \otimes \mathcal{H}^\pm$, the case of \mathcal{K}^0 being more subtle, and, as a consequence of these embeddings, we obtain projectors of \mathcal{K} onto its irreducible components (Theorem 12, Corollary 14 and Theorem 15). In Section 7 we give a new derivation of the identification of the one-loop Feynman diagram with the integral kernels of the projection operators

$$\mathcal{P}^+ : \mathcal{H}^+ \otimes \mathcal{H}^+ \twoheadrightarrow \mathcal{K}^+ \hookrightarrow \mathcal{H}^+ \otimes \mathcal{H}^+ \quad \text{and} \quad \mathcal{P}^- : \mathcal{H}^- \otimes \mathcal{H}^- \twoheadrightarrow \mathcal{K}^- \hookrightarrow \mathcal{H}^- \otimes \mathcal{H}^-$$

(cf. [FL1]). In Section 8 we realize the spaces \mathcal{K}^\pm , \mathcal{K}^0 as well as the results of Section 6 in the setting of the Minkowski space \mathbb{M} . In the second part of the paper, Section 9, we introduce the anti de Sitter deformation of the space of quaternions \mathbb{H} together with the conformal Laplacian

(10). Then in Section 10 we describe the action of the conformal algebra $\mathfrak{so}(1, 5)$ on the kernel of the conformal Laplacian. In Section 11 we find simple extensions of the elements of the kernel to $\mathbb{R}^{1,4}$ as solutions of the wave equation. In Section 12 we introduce a space \mathcal{H}_μ consisting of the K -finite elements of the kernel of the conformal Laplacian. Similarly to (7), we have a decomposition into irreducible components $\mathcal{H}_\mu = \mathcal{H}_\mu^+ \oplus \mathcal{H}_\mu^-$. In Section 13 we prove an analogue of the Poisson formula for the solutions of $\tilde{\square}_\mu \varphi = 0$ (Theorem 32). In Section 14 we factor the conformal Laplacian as a product of two Dirac-type operators (11). In Sections 15 and 16 we proceed to study deformed quaternionic regular functions. We prove analogues of the Cauchy's Theorem and the Cauchy-Fueter formula in the deformed setting (Corollary 38 and Theorem 41). Finally, in Section 17 we introduce a deformation \mathcal{H}_μ of the space \mathcal{H} associated with the second order pole, similarly to (8), decompose it into a direct sum $\mathcal{H}_\mu^- \oplus \mathcal{H}_\mu^0 \oplus \mathcal{H}_\mu^+$ and obtain projectors onto these direct summands.

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2 Preliminaries

We recall some notations from [FL1]. Let $\mathbb{H}_\mathbb{C}$ denote the space of complexified quaternions: $\mathbb{H}_\mathbb{C} = \mathbb{H} \otimes \mathbb{C}$, it can be identified with the algebra of 2×2 complex matrices:

$$\mathbb{H}_\mathbb{C} = \mathbb{H} \otimes \mathbb{C} \simeq \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}; z_{ij} \in \mathbb{C} \right\} = \left\{ Z = \begin{pmatrix} z^0 - iz^3 & -iz^1 - z^2 \\ -iz^1 + z^2 & z^0 + iz^3 \end{pmatrix}; z^k \in \mathbb{C} \right\}.$$

For $Z \in \mathbb{H}_\mathbb{C}$, we write

$$N(Z) = \det \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = z_{11}z_{22} - z_{12}z_{21} = (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2$$

and think of it as (the square of) the norm of Z . We denote by $\mathbb{H}_\mathbb{C}^\times$ the group of invertible complexified quaternions:

$$\mathbb{H}_\mathbb{C}^\times = \{Z \in \mathbb{H}_\mathbb{C}; N(Z) \neq 0\}.$$

Clearly, $\mathbb{H}_\mathbb{C}^\times \simeq GL(2, \mathbb{C})$. We realize $U(2)$ as a subgroup of $\mathbb{H}_\mathbb{C}^\times$:

$$U(2) = \{Z \in \mathbb{H}_\mathbb{C}; Z^* = Z^{-1}\},$$

where Z^* denotes the complex conjugate transpose of a complex matrix Z . For $R > 0$, we set

$$U(2)_R = \{RZ; Z \in U(2)\}$$

and orient it as in [FL1] so that $\int_{U(2)_R} \frac{dV}{N(Z)^2} = -2\pi^3 i$, where dV is a holomorphic 4-form defined by

$$dV = dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3 = \frac{1}{4} dz_{11} \wedge dz_{12} \wedge dz_{21} \wedge dz_{22}.$$

Recall that a group $GL(2, \mathbb{H}_\mathbb{C}) \simeq GL(4, \mathbb{C})$ acts on $\mathbb{H}_\mathbb{C}$ by fractional linear (or conformal) transformations:

$$h : Z \mapsto (aZ + b)(cZ + d)^{-1} = (a' - Zc')^{-1}(-b' + Zd'), \quad Z \in \mathbb{H}_\mathbb{C}, \quad (13)$$

where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C})$ and $h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

For convenience we recall Lemmas 10 and 61 from [FL1]:

Lemma 1. For $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ with $h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, let $\tilde{Z} = (aZ + b)(cZ + d)^{-1}$ and $\tilde{W} = (aW + b)(cW + d)^{-1}$. Then

$$\begin{aligned} (\tilde{Z} - \tilde{W}) &= (a' - Wc')^{-1} \cdot (Z - W) \cdot (cZ + d)^{-1} \\ &= (a' - Zc')^{-1} \cdot (Z - W) \cdot (cW + d)^{-1}. \end{aligned}$$

Lemma 2. Let $d\tilde{V}$ denote the pull-back of dV under the map $Z \mapsto (aZ + b)(cZ + d)^{-1}$, where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ and $h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then

$$dV = N(cZ + d)^2 \cdot N(a' - Zc')^2 d\tilde{V}.$$

We often use the matrix coefficient functions of $SU(2)$ described by equation (27) of [FL1] (cf. [V]):

$$t_{n\underline{m}}^l(Z) = \frac{1}{2\pi i} \oint (sz_{11} + z_{21})^{l-m} (sz_{12} + z_{22})^{l+m} s^{-l+n} \frac{ds}{s}, \quad \begin{aligned} l &= 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \\ m, n &\in \mathbb{Z} + l, \\ -l &\leq m, n \leq l, \end{aligned} \quad (14)$$

$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}$, the integral is taken over a loop in \mathbb{C} going once around the origin in the counterclockwise direction. We regard these functions as polynomials on $\mathbb{H}_{\mathbb{C}}$. For future use we state the multiplicativity property of matrix coefficients

$$t_{m\underline{n}}^l(Z_1 Z_2) = \sum_{j=-l}^l t_{m\underline{j}}^l(Z_1) \cdot t_{j\underline{n}}^l(Z_2). \quad (15)$$

It is also useful to recall that

$$t_{m\underline{n}}^l(Z^{-1}) = t_{m\underline{n}}^l(Z^+) \cdot N(Z)^{-2l} \quad \text{is proportional to} \quad t_{-n\underline{-m}}^l(Z) \cdot N(Z)^{-2l}.$$

As in Section 2 of [FL2], we consider the space of \mathbb{C} -valued functions on $\mathbb{H}_{\mathbb{C}}$ (possibly with singularities) which are holomorphic with respect to the complex variables z^0, z^1, z^2, z^3 or $z_{11}, z_{12}, z_{21}, z_{22}$ and harmonic, i.e. satisfying $\square\varphi = 0$, where

$$\square = \frac{\partial^2}{(\partial z^0)^2} + \frac{\partial^2}{(\partial z^1)^2} + \frac{\partial^2}{(\partial z^2)^2} + \frac{\partial^2}{(\partial z^3)^2} = 4 \left(\frac{\partial^2}{\partial z_{11} \partial z_{22}} - \frac{\partial^2}{\partial z_{12} \partial z_{21}} \right).$$

We denote this space by $\tilde{\mathcal{H}}$. Then the conformal group $GL(2, \mathbb{H}_{\mathbb{C}}) \simeq GL(4, \mathbb{C})$ acts on $\tilde{\mathcal{H}}$ by two slightly different actions:

$$\begin{aligned} \pi_l^0(h) : \varphi(Z) &\mapsto (\pi_l^0(h)\varphi)(Z) = \frac{1}{N(cZ + d)} \cdot \varphi((aZ + b)(cZ + d)^{-1}), \\ \pi_r^0(h) : \varphi(Z) &\mapsto (\pi_r^0(h)\varphi)(Z) = \frac{1}{N(a' - Zc')} \cdot \varphi((a' - Zc')^{-1}(-b' + Zd')), \end{aligned}$$

where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ and $h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. We have

$$(aZ + b)(cZ + d)^{-1} = (a' - Zc')^{-1}(-b' + Zd'), \quad \forall Z \in \mathbb{H}_{\mathbb{C}},$$

and these two actions coincide on $SL(2, \mathbb{H}_{\mathbb{C}}) \simeq SL(4, \mathbb{C})$, which is defined as the connected Lie subgroup of $GL(2, \mathbb{H}_{\mathbb{C}})$ with Lie algebra

$$\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}}) = \{x \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}); \operatorname{Re}(\operatorname{Tr} x) = 0\} \simeq \mathfrak{sl}(4, \mathbb{C}).$$

We introduce two spaces of harmonic polynomials:

$$\mathcal{H}^+ = \tilde{\mathcal{H}} \cap \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}],$$

$$\mathcal{H} = \tilde{\mathcal{H}} \cap \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, N(Z)^{-1}]$$

and the space of harmonic polynomials regular at infinity:

$$\mathcal{H}^- = \{\varphi \in \tilde{\mathcal{H}}; N(Z)^{-1} \cdot \varphi(Z^{-1}) \in \mathcal{H}^+\}.$$

Then

$$\begin{aligned} \mathcal{H} &= \mathcal{H}^- \oplus \mathcal{H}^+, \\ \mathcal{H}^+ &= \text{Span}\{t_{n\underline{m}}^l(Z)\}, \\ \mathcal{H}^- &= \text{Span}\{t_{n\underline{m}}^l(Z) \cdot N(Z)^{-(2l+1)}\}. \end{aligned}$$

In particular, there are no homogeneous harmonic functions in $\mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, N(Z)^{-1}]$ of degree -1 . Differentiating the actions π_l^0 and π_r^0 , we obtain actions of $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}) \simeq \mathfrak{gl}(4, \mathbb{C})$ which preserve the spaces \mathcal{H} , \mathcal{H}^- and \mathcal{H}^+ . By abuse of notation, we denote these Lie algebra actions by π_l^0 and π_r^0 respectively. They are described in Subsection 3.2 of [FL2].

By Theorem 28 in [FL1], for each $R > 0$, we have a bilinear pairing between (π_l^0, \mathcal{H}) and (π_r^0, \mathcal{H}) :

$$(\varphi_1, \varphi_2)_R = \frac{1}{2\pi^2} \int_{S_R^3} (\widetilde{\deg \varphi_1})(Z) \cdot \varphi_2(Z) \frac{dS}{R}, \quad \varphi_1, \varphi_2 \in \mathcal{H}, \quad (16)$$

where $S_R^3 \subset \mathbb{H}$ is the three-dimensional sphere of radius R centered at the origin

$$S_R^3 = \{X \in \mathbb{H}; N(X) = R^2\},$$

dS denotes the usual Euclidean volume element on S_R^3 , and $\widetilde{\deg}$ denotes the degree operator plus identity:

$$\widetilde{\deg} f = f + \deg f = f + z_{11} \frac{\partial f}{\partial z_{11}} + z_{12} \frac{\partial f}{\partial z_{12}} + z_{21} \frac{\partial f}{\partial z_{21}} + z_{22} \frac{\partial f}{\partial z_{22}}.$$

When this pairing is restricted to $\mathcal{H}^+ \times \mathcal{H}^-$, it is $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant, independent of the choice of $R > 0$, non-degenerate and antisymmetric

$$(\varphi_1, \varphi_2)_R = -(\varphi_2, \varphi_1)_R, \quad \varphi_1 \in \mathcal{H}^+, \varphi_2 \in \mathcal{H}^-.$$

We have the following orthogonality relations with respect to the pairing (16):

$$(t_{n'\underline{m}'}^{l'}(Z), t_{m\underline{n}}^l(Z^{-1}) \cdot N(Z)^{-1})_R = -(t_{m\underline{n}}^l(Z^{-1}) \cdot N(Z)^{-1}, t_{n'\underline{m}'}^{l'}(Z))_R = \delta_{ll'} \delta_{mm'} \delta_{nn'}, \quad (17)$$

where the indices l, m, n are $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, $m, n \in \mathbb{Z} + l$, $-l \leq m, n \leq l$ and similarly for l', m', n' .

Let $\tilde{\mathcal{K}}$ denote the space of \mathbb{C} -valued functions on $\mathbb{H}_{\mathbb{C}}$ (possibly with singularities) which are holomorphic with respect to the complex variables $z_{11}, z_{12}, z_{21}, z_{22}$. (There are no differential equations imposed on functions in $\tilde{\mathcal{K}}$ whatsoever.) We recall the action of $GL(2, \mathbb{H}_{\mathbb{C}})$ on $\tilde{\mathcal{K}}$ given by equation (49) in [FL1]:

$$\rho_1(h) : f(Z) \mapsto (\rho_1(h)f)(Z) = \frac{f((aZ + b)(cZ + d)^{-1})}{N(cZ + d) \cdot N(a' - Zc')},$$

where $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ and $h^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. We have a natural $GL(2, \mathbb{H}_{\mathbb{C}})$ -equivariant multiplication map

$$M : (\pi_l^0, \tilde{\mathcal{H}}) \otimes (\pi_r^0, \tilde{\mathcal{H}}) \rightarrow (\rho_1, \tilde{\mathcal{K}})$$

which is determined on pure tensors by

$$M(\varphi_1(Z_1) \otimes \varphi_2(Z_2)) = (\varphi_1 \cdot \varphi_2)(Z), \quad \varphi_1, \varphi_2 \in \tilde{\mathcal{H}}.$$

Differentiating the ρ_1 -action, we obtain an action of $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}) \simeq \mathfrak{gl}(4, \mathbb{C})$. For convenience we recall Lemma 68 from [FL1].

Lemma 3. *Let $\partial = \begin{pmatrix} \partial_{11} & \partial_{21} \\ \partial_{12} & \partial_{22} \end{pmatrix} = \frac{1}{2} \nabla$, where $\partial_{ij} = \frac{\partial}{\partial z_{ij}}$. The Lie algebra action ρ_1 of $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ on $\tilde{\mathcal{K}}$ is given by*

$$\begin{aligned} \rho_1 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : f &\mapsto \text{Tr}(A \cdot (-Z \cdot \partial f - f)) \\ \rho_1 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : f &\mapsto \text{Tr}(B \cdot (-\partial f)) \\ \rho_1 \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : f &\mapsto \text{Tr}(C \cdot (Z \cdot (\partial f) \cdot Z + 2Zf)) = \text{Tr}(C \cdot (Z \cdot \partial(Zf))) \\ \rho_1 \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} : f &\mapsto \text{Tr}(D \cdot ((\partial f) \cdot Z + f)) = \text{Tr}(D \cdot (\partial(Zf) - f)). \end{aligned}$$

This lemma implies that $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ preserves the spaces

$$\begin{aligned} \mathcal{K}^+ &= \{\text{polynomial functions on } \mathbb{H}_{\mathbb{C}}\} = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}] \quad \text{and} \\ \mathcal{K} &= \{\text{polynomial functions on } \mathbb{H}_{\mathbb{C}}^{\times}\} = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, N(Z)^{-1}]. \end{aligned}$$

Define

$$\mathcal{K}^- = \left\{ f \in \mathcal{K}; \rho_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f(Z) = N(Z)^{-2} \cdot f(Z^{-1}) \in \mathcal{K}^+ \right\},$$

this is another $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant space. Comparing this with Definition 16 in [FL1], we can say that \mathcal{K}^- consists of those elements of \mathcal{K} that are regular at infinity according to the ρ_1 -action of $GL(2, \mathbb{H}_{\mathbb{C}})$. Note that $\mathcal{K}^- \oplus \mathcal{K}^+$ is a proper subspace of \mathcal{K} .

Next we describe an invariant bilinear pairing on \mathcal{K} . Recall Proposition 69 from [FL1]:

Proposition 4. *The representation (ρ_1, \mathcal{K}) of $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ has a non-degenerate symmetric bilinear pairing*

$$\langle f_1, f_2 \rangle = \frac{i}{2\pi^3} \int_{U(2)_R} f_1(Z) \cdot f_2(Z) dV, \quad f_1, f_2 \in \mathcal{K}, \quad (18)$$

where $R > 0$. This bilinear pairing is $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant and independent of the choice of $R > 0$.

We have the following orthogonality relations with respect to the pairing (18):

$$\langle t_{n' \underline{m}'}^{l'}(Z) \cdot N(Z)^{k'}, t_{m \underline{n}}^l(Z^{-1}) \cdot N(Z)^{-k-2} \rangle = \frac{1}{2l+1} \delta_{kk'} \delta_{ll'} \delta_{mm'} \delta_{nn'}, \quad (19)$$

where the indices k, l, m, n are $k \in \mathbb{Z}$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, $m, n \in \mathbb{Z} + l$, $-l \leq m, n \leq l$ and similarly for k', l', m', n' .

We know from [JV] and [FL1] that the representations (ρ_1, \mathcal{K}^+) and (ρ_1, \mathcal{K}^-) are \mathbb{C} -linear dual to each other with respect to (18), irreducible when restricted to $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$ and possess

inner products which make them unitary representations of the real form $\mathfrak{su}(2, 2)$ of $\mathfrak{sl}(2, \mathbb{H}_{\mathbb{C}})$, where we regard $\mathfrak{su}(2, 2)$ and $\mathfrak{u}(2, 2)$ as subalgebras of $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ as in (20).

We often regard the group $U(2, 2)$ as a subgroup of $GL(2, \mathbb{H}_{\mathbb{C}})$ as described in Subsection 3.5 of [FL1]. That is

$$U(2, 2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}}); \begin{array}{l} a^*a = 1 + c^*c \\ d^*d = 1 + b^*b \\ a^*b = c^*d \end{array} \right\}.$$

The Lie algebra of $U(2, 2)$ is

$$\mathfrak{u}(2, 2) = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}); A, B, D \in \mathbb{H}_{\mathbb{C}}, A = -A^*, D = -D^* \right\}. \quad (20)$$

The maximal compact subgroup of $U(2, 2)$ is

$$U(2) \times U(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}}); a, d \in \mathbb{H}_{\mathbb{C}}, a^*a = d^*d = 1 \right\}. \quad (21)$$

The group $U(2, 2)$ acts on $\mathbb{H}_{\mathbb{C}}$ by fractional linear transformations (13) preserving $U(2) \subset \mathbb{H}_{\mathbb{C}}$ and open domains

$$\mathbb{D}^+ = \{Z \in \mathbb{H}_{\mathbb{C}}; ZZ^* < 1\}, \quad \mathbb{D}^- = \{Z \in \mathbb{H}_{\mathbb{C}}; ZZ^* > 1\},$$

where the inequalities $ZZ^* < 1$ and $ZZ^* > 1$ mean that the matrix $ZZ^* - 1$ is negative and positive definite respectively. The sets \mathbb{D}^+ and \mathbb{D}^- both have $U(2)$ as the Shilov boundary.

Similarly, for each $R > 0$ we can define a conjugate of $U(2, 2)$

$$U(2, 2)_R = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} U(2, 2) \begin{pmatrix} R^{-1} & 0 \\ 0 & 1 \end{pmatrix} \subset GL(2, \mathbb{H}_{\mathbb{C}}).$$

Each group $U(2, 2)_R$ is a real form of $GL(2, \mathbb{H}_{\mathbb{C}})$, preserves $U(2)_R$ and open domains

$$\mathbb{D}_R^+ = \{Z \in \mathbb{H}_{\mathbb{C}}; ZZ^* < R^2\}, \quad \mathbb{D}_R^- = \{Z \in \mathbb{H}_{\mathbb{C}}; ZZ^* > R^2\}. \quad (22)$$

These sets \mathbb{D}_R^+ and \mathbb{D}_R^- both have $U(2)_R$ as the Shilov boundary.

3 K -type Basis of (ρ_1, \mathfrak{K})

In this section we describe a convenient basis of (ρ_1, \mathfrak{K}) consisting of K -types for the maximal compact subgroup $U(2) \times U(2)$ of $U(2, 2)$.

Proposition 5. *The functions*

$$t_{n\underline{m}}^l(Z) \cdot N(Z)^k, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad m, n = -l, -l+1, \dots, l, \quad k = 0, 1, 2, \dots, \quad (23)$$

form a vector space basis of $\mathfrak{K}^+ = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}]$.

Proof. Clearly, the functions $t_{n\underline{m}}^l(Z) \cdot N(Z)^k$ are polynomials. From the orthogonality relations (19) it follows that they are linearly independent. It remains to show that they span all of \mathfrak{K}^+ . We can do that by comparing the dimensions of the subspaces homogeneous functions of degree d in \mathfrak{K}^+ and the space spanned by (23).

The number of monomials $(z_{11})^{\alpha_{11}}(z_{12})^{\alpha_{12}}(z_{21})^{\alpha_{21}}(z_{22})^{\alpha_{22}}$ in \mathcal{H}^+ with $\alpha_{11} + \alpha_{12} + \alpha_{21} + \alpha_{22} = d$ is

$$\binom{d+3}{3} = \frac{(d+3)(d+2)(d+1)}{6}. \quad (24)$$

On the other hand, for k and l fixed, there are exactly $(2l+1)^2$ basis elements (23) and they are all homogeneous of degree $2l+2k$. Therefore, the dimension of the subspace of homogeneous functions of degree d inside the span of (23) is

$$(d+1)^2 + (d-1)^2 + (d-3)^2 + \dots \quad (25)$$

Finally, it is easy to show by induction that (24) and (25) are in fact equal. \square

We conclude this section with a decomposition of (ρ_1, \mathcal{H}) into K -types.

Corollary 6. *The functions*

$$t_{n\underline{m}}^l(Z) \cdot N(Z)^k, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad m, n = -l, -l+1, \dots, l, \quad k \in \mathbb{Z},$$

form a vector space basis of $\mathcal{H} = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, N(Z)^{-1}]$.

Proof. The functions $t_{n\underline{m}}^l(Z) \cdot N(Z)^k$ are linearly independent by (19) and by Proposition 5 span the entire space \mathcal{H} . \square

4 Irreducible Components of (ρ_1, \mathcal{H})

In this section we decompose (ρ_1, \mathcal{H}) into irreducible components, identify these irreducible components as images of multiplication maps and describe their unitary structures.

Theorem 7. *The representation (ρ_1, \mathcal{H}) of $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ has the following decomposition into irreducible components:*

$$(\rho_1, \mathcal{H}) = (\rho_1, \mathcal{H}^-) \oplus (\rho_1, \mathcal{H}^0) \oplus (\rho_1, \mathcal{H}^+),$$

where

$$\begin{aligned} \mathcal{H}^+ &= \mathbb{C} - \text{span of } \{t_{n\underline{m}}^l(Z) \cdot N(Z)^k; k \geq 0\}, \\ \mathcal{H}^- &= \mathbb{C} - \text{span of } \{t_{n\underline{m}}^l(Z) \cdot N(Z)^k; k \leq -(2l+2)\}, \\ \mathcal{H}^0 &= \mathbb{C} - \text{span of } \{t_{n\underline{m}}^l(Z) \cdot N(Z)^k; -(2l+1) \leq k \leq -1\}. \end{aligned}$$

Proof. Note that the basis elements (23) consist of functions of the kind

$$f_l(Z) \cdot N(Z)^k, \quad \square f_l(Z) = 0, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad k \in \mathbb{Z},$$

where the functions $f_l(Z)$ range over a basis of harmonic functions which are polynomials of degree $2l$. Recall that we consider $U(2) \times U(2)$ as a subgroup of $GL(2, \mathbb{H}_{\mathbb{C}})$ via (21). For k and l fixed, these functions span an irreducible representation of $U(2) \times U(2)$, which – when restricted to $SU(2) \times SU(2)$ – becomes isomorphic to $V_l \boxtimes V_l$, where V_l denotes the irreducible representation of $SU(2)$ of dimension $2l+1$.

To determine the effect of matrices of the kind $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ with $B \in \mathbb{H}_{\mathbb{C}}$, we use Lemma 3 describing their action and compute

$$\partial(f_l(Z) \cdot N(Z)^k) = \partial f_l \cdot N(Z)^k + kZ^+ f_l \cdot N(Z)^{k-1}.$$

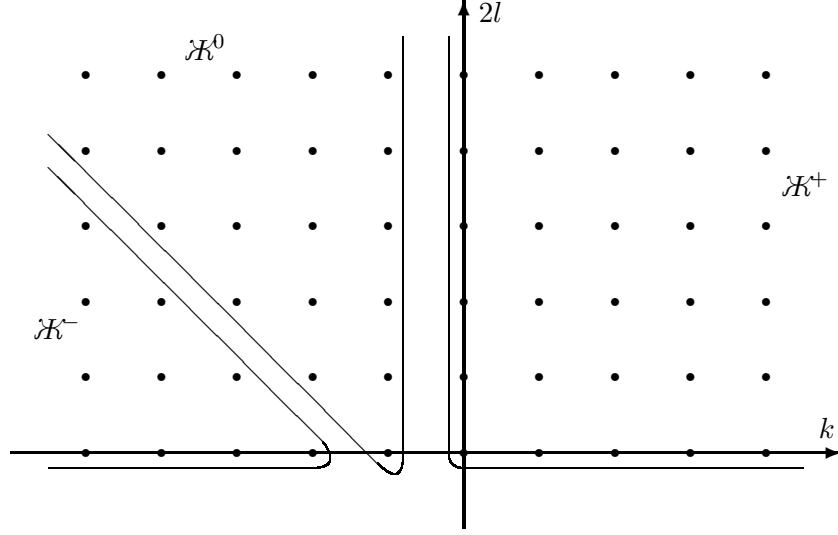


Figure 2: Decomposition of (ρ_1, \mathcal{H}) into irreducible components

By direct computation we have:

$$\partial f_l \cdot N(Z) = Z^+ \deg f_l - Z^+ \cdot (\partial^+ f_l) \cdot Z^+ = 2lZ^+ f_l - Z^+ \cdot (\partial^+ f_l) \cdot Z^+,$$

$$\square(Z^+ f_l) = Z^+ \square f_l + 4\partial f_l \quad \text{and} \quad \square(N(Z) \cdot g) = N(Z) \cdot \square g + 4(\deg + 2)g.$$

Hence we can write

$$Z^+ f_l = \left(Z^+ f_l - \frac{\partial f_l \cdot N(Z)}{2l+1} \right) + \frac{\partial f_l \cdot N(Z)}{2l+1} = \frac{Z^+ \cdot (\partial^+ f_l) \cdot Z^+ + Z^+ f_l}{2l+1} + \frac{\partial f_l \cdot N(Z)}{2l+1} \quad (26)$$

and

$$\partial(f_l(Z) \cdot N(Z)^k) = \frac{2l+k+1}{2l+1} \partial f_l \cdot N(Z)^k + \frac{k}{2l+1} (Z^+ \cdot (\partial^+ f_l) \cdot Z^+ + Z^+ f_l) \cdot N(Z)^{k-1} \quad (27)$$

with ∂f_l and $Z^+ \cdot (\partial^+ f_l) \cdot Z^+ + Z^+ f_l$ being harmonic and having degrees $2l-1$ and $2l+1$ respectively.

Next we determine the effect of matrices of the kind $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ with $C \in \mathbb{H}_{\mathbb{C}}$. Again, we use Lemma 3 and compute

$$Z \cdot \partial(f_l \cdot N(Z)^k) \cdot Z + 2Z f_l \cdot N(Z)^k = Z \cdot (\partial f_l) \cdot Z \cdot N(Z)^k + (k+2)Z f_l \cdot N(Z)^k.$$

Conjugating (26) we see that

$$Z f_l = \frac{Z \cdot (\partial f_l) \cdot Z + Z f_l}{2l+1} + \frac{\partial^+ f_l \cdot N(Z)}{2l+1}.$$

Therefore,

$$\begin{aligned} Z \cdot \partial(f_l \cdot N(Z)^k) \cdot Z + 2Z f_l \cdot N(Z)^k \\ = \frac{2l+k+2}{2l+1} (Z \cdot (\partial f_l) \cdot Z + Z f_l) \cdot N(Z)^k + \frac{k+1}{2l+1} \partial^+ f_l \cdot N(Z)^{k+1} \end{aligned} \quad (28)$$

with $Z \cdot (\partial f_l) \cdot Z + Z f_l$ and $\partial^+ f_l$ being harmonic and having degrees $2l+1$ and $2l-1$ respectively.

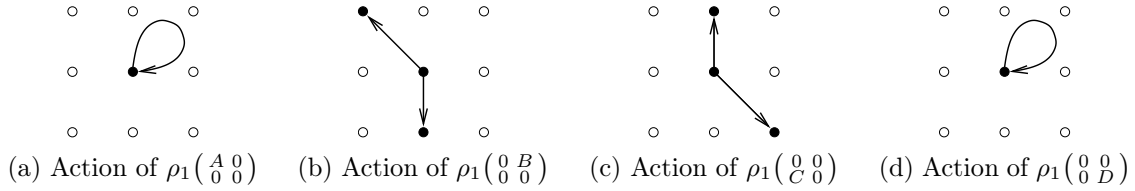


Figure 3

The actions of $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ are illustrated in Figure 3. In the diagram describing $\rho_1 \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ the vertical arrow disappears if $l = 0$ or $2l + k + 1 = 0$ and the diagonal arrow disappears if $k = 0$. Similarly, in the diagram describing $\rho_1 \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ the vertical arrow disappears if $2l + k + 2 = 0$ and the diagonal arrow disappears if $k = -1$ or $l = 0$. This proves that \mathcal{K}^+ , \mathcal{K}^- and \mathcal{K}^0 are $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -invariant subspaces of \mathcal{K} . Note that

$$\mathrm{Tr}(Z \cdot \partial f + f) = \mathrm{Tr} \begin{pmatrix} z_{11}\partial_{11}f + z_{12}\partial_{12}f + f & * & * & * \\ * & * & * & * \\ * & * & * & * \\ z_{21}\partial_{21}f + z_{22}\partial_{22}f + f & & & \end{pmatrix} = (\deg + 2)f,$$

hence $Z \cdot (\partial f_l) \cdot Z + Z f_l = (Z \cdot \partial f_l + f_l) \cdot Z$ and its conjugate $Z^+ \cdot (\partial^+ f_l) \cdot Z^+ + Z^+ f_l$ are never zero. It follows from (27) and (28) that the subrepresentations (ρ_1, \mathcal{K}^+) , $(\rho_1, \mathcal{K}_2^-)$, (ρ_1, \mathcal{K}^0) are irreducible with respect to ρ_1 -action of $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$. \square

Our next task is to identify the images under the natural $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariant multiplication maps:

$$M : (\pi_l^0, \mathcal{H}^{\pm}) \otimes (\pi_r^0, \mathcal{H}^{\pm}) \rightarrow (\rho_1, \mathcal{K}) \quad (29)$$

sending pure tensors

$$\varphi_1(Z_1) \otimes \varphi_2(Z_2) \mapsto (\varphi_1 \cdot \varphi_2)(Z).$$

Lemma 8. *Under the multiplication maps $(\pi_l^0, \mathcal{H}^{\pm}) \otimes (\pi_r^0, \mathcal{H}^{\pm}) \rightarrow (\rho_1, \mathcal{K})$,*

1. *The image of $\mathcal{H}^+ \otimes \mathcal{H}^+$ in \mathcal{K} is \mathcal{K}^+ ;*
2. *The image of $\mathcal{H}^- \otimes \mathcal{H}^-$ in \mathcal{K} is \mathcal{K}^- ;*
3. *The image of $\mathcal{H}^- \otimes \mathcal{H}^+$ in \mathcal{K} is \mathcal{K}^0 .*

Proof. Note that the space \mathcal{H}^+ consists of harmonic polynomials. The product of two polynomials is another polynomial, hence the image of $\mathcal{H}^+ \otimes \mathcal{H}^+$ lies in \mathcal{K}^+ . Since (ρ_1, \mathcal{K}) is irreducible, the image is all of \mathcal{K}^+ .

Applying $(\pi_l^0 \otimes \pi_r^0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to the left hand side of $\mathcal{H}^+ \otimes \mathcal{H}^+ \rightarrow \mathcal{K}^+$ and $\rho_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to the right hand side, we see that the image of $\mathcal{H}^- \otimes \mathcal{H}^-$ is \mathcal{K}^- .

Let us denote by J the image of $\mathcal{H}^- \otimes \mathcal{H}^+$ in \mathcal{K} . Clearly, J contains the function $N(Z)^{-1}$, which generates \mathcal{K}^0 . Hence $\mathcal{K}^0 \subset J$. It remains to show that $J \subset \mathcal{K}^0$. By Theorem 7, if $\mathcal{K}^0 \subsetneq J$, then J also contains \mathcal{K}^+ or \mathcal{K}^- and hence functions $N(Z)^k$ with $k \neq -1$. Thus it is sufficient to prove that J cannot contain $N(Z)^k$ with $k \neq -1$.

By construction, J is spanned by

$$N(Z)^{-(2l+1)} \cdot t_{n\underline{m}}^l(Z) \cdot t_{n'\underline{m}'}^{l'}(Z). \quad (30)$$

Note that if V_l and $V_{l'}$ are two irreducible representations of $SU(2)$ of dimensions $2l + 1$ and $2l' + 1$ respectively, then their tensor product contains a copy of the trivial representation if and only if $l = l'$. This means that a linear combination of the functions (30) can express $N(Z)^k$ only if $l = l'$. But then the homogeneity degree of (30) is -2 . Therefore, $N(Z)^k \notin J$ if $k \neq -1$. \square

As we have mentioned, the representations (ρ_1, \mathcal{H}^+) and (ρ_1, \mathcal{H}^-) are \mathbb{C} -linear dual to each other with respect to (18). On the other hand, the \mathbb{C} -linear dual of (ρ_1, \mathcal{H}^0) with respect to (18) is (ρ_1, \mathcal{H}^0) itself. We conclude this section with an explicit description of the unitary structures on (ρ_1, \mathcal{H}^+) , (ρ_1, \mathcal{H}^-) and (ρ_1, \mathcal{H}^0) . Define

$$(f_1, f_2) = \frac{i}{2\pi^3} \int_{U(2)} f_1(Z) \cdot \overline{f_2(Z)} \frac{dV}{N(Z)^2}, \quad f_1, f_2 \in \mathcal{H}. \quad (31)$$

This pairing is an inner product.

Proposition 9. *The restrictions of (ρ_1, \mathcal{H}^+) , (ρ_1, \mathcal{H}^-) and (ρ_1, \mathcal{H}^0) to $\mathfrak{u}(2, 2)$ are unitary with respect to the inner product (31).*

Proof. We only need to prove that the pairing (31) is $\mathfrak{u}(2, 2)$ -invariant. It is enough to show that, for all $h \in U(2, 2)$ sufficiently close to the identity element, we have

$$(f_1, f_2) = (\rho_1(h)f_1, \rho_1(h)f_2), \quad f_1, f_2 \in \mathcal{H}.$$

If $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2, 2)$, then $h = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}$. (If $Z \in \mathbb{H}_{\mathbb{C}}$, $Z^* \in \mathbb{H}_{\mathbb{C}}$ denotes the matrix adjoint of Z under the standard identification of $\mathbb{H}_{\mathbb{C}}$ with 2×2 complex matrices, see [FL1] for details.) Writing $\tilde{Z} = (aZ + b)(cZ + d)^{-1}$ and using Lemma 2 together with the fact that $U(2, 2)$ preserves $U(2) = \{Z \in \mathbb{H}_{\mathbb{C}}; Z^* = Z^{-1}\}$ we obtain:

$$\begin{aligned} & -2\pi^3 i \cdot (\rho_1(h)f_1, \rho_1(h)f_2) \\ &= \int_{Z \in U(2)} \frac{f_1(\tilde{Z})}{N(cZ + d) \cdot N(a^* + Zb^*)} \cdot \overline{\frac{f_2(\tilde{Z})}{N(cZ + d) \cdot N(a^* + Zb^*)}} \frac{dV}{N(Z)^2} \\ &= \int_{\tilde{Z} \in U(2)} f_1(\tilde{Z}) \cdot \overline{f_2(\tilde{Z})} \frac{N(c^* + Zd^*) \cdot N(aZ + b)}{N(cZ + d) \cdot N(a^* + Zb^*) \cdot N(Z)^2} \frac{dV}{N(\tilde{Z})^2} \\ &= \int_{\tilde{Z} \in U(2)} f_1(\tilde{Z}) \cdot \overline{f_2(\tilde{Z})} \frac{dV}{N(\tilde{Z})^2} = -2\pi^3 i \cdot (f_1, f_2). \end{aligned}$$

□

5 Formal Calculation of the Reproducing Kernel for (ρ_1, \mathcal{H}^0)

In [FL1], Proposition 27, we computed the reproducing kernels for (ρ_1, \mathcal{H}^+) and (ρ_1, \mathcal{H}^-) by finding expansions for $\frac{1}{N(Z-W)^2}$ in terms of basis functions (23). In both cases the reproducing kernel is $\frac{1}{N(Z-W)^2}$, but one gets different results depending on whether ZW^{-1} lies in \mathbb{D}^+ or \mathbb{D}^- :

Proposition 10 (Proposition 27, [FL1]). *We have the following matrix coefficient expansions*

$$\frac{1}{N(Z-W)^2} = \sum_{k,l,m,n} (2l+1)t_{mn}^l(Z^{-1}) \cdot N(Z)^{-k-2} \cdot t_{nm}^l(W) \cdot N(W)^k$$

which converges pointwise absolutely in the region $\{(Z, W) \in \mathbb{H}_{\mathbb{C}}^{\times} \times \mathbb{H}_{\mathbb{C}}; WZ^{-1} \in \mathbb{D}^+\}$, and

$$\frac{1}{N(Z-W)^2} = \sum_{k,l,m,n} (2l+1)t_{mn}^l(Z) \cdot N(Z)^k \cdot t_{nm}^l(W^{-1}) \cdot N(W)^{-k-2}$$

which converges pointwise absolutely in the region $\{(Z, W) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}^{\times}; ZW^{-1} \in \mathbb{D}^+\}$. The sums are taken first over all $m, n = -l, -l+1, \dots, l$, then over $k = 0, 1, 2, 3, \dots$ and $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

In this section we formally compute the reproducing kernel for (ρ_1, \mathcal{H}^0) . There are some issues with convergence that require justification, but it is nice to see that this formally computed kernel agrees with the formula for a projector onto \mathcal{H}^0 that will be obtained in the next section (Theorem 15).

Recall that \mathcal{H}^0 is the \mathbb{C} -span of $\{t_{n\underline{m}}^l(Z) \cdot N(Z)^k; -(2l+1) \leq k \leq -1\}$. In light of the orthogonality relations (19) we would like to compute the series

$$\sum_{k,l,m,n} (2l+1)t_{n\underline{m}}^l(Z^{-1}) \cdot N(Z)^{-k-2} \cdot t_{m\underline{n}}^l(W) \cdot N(W)^k, \quad (32)$$

the sum is being taken over all $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, $m, n \in \mathbb{Z} + l$ with $m, n = -l, -l+1, \dots, l$ and $-(2l+1) \leq k \leq -1$. By the multiplicativity property of matrix coefficients (15), (32) equals

$$\sum_{k,l,n} \frac{2l+1}{N(Z)^2} \cdot t_{n\underline{n}}^l(Z^{-1}W) \cdot N(Z^{-1}W)^k = \sum_{l,n} \frac{2l+1}{N(Z)^2} \cdot t_{n\underline{n}}^l(Z^{-1}W) \cdot \frac{N(Z^{-1}W)^{-(2l+1)} - 1}{1 - N(Z^{-1}W)}.$$

Assume further that $Z^{-1}W$ can be diagonalized as $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1 \neq \lambda_2$. This is allowed since the set of matrices with different eigenvalues is dense in $\mathbb{H}_{\mathbb{C}}$. Then the sum $\sum_n t_{n\underline{n}}^l(Z^{-1}W)$ is just the character $\chi_l(Z^{-1}W)$ of the irreducible representation of $GL(2, \mathbb{C})$ of dimension $2l+1$ and equals $\frac{\lambda_1^{2l+1} - \lambda_2^{2l+1}}{\lambda_1 - \lambda_2}$. Hence (32) is equal to

$$\begin{aligned} \sum_l \frac{2l+1}{N(Z)^2} \cdot \frac{\lambda_1^{2l+1} - \lambda_2^{2l+1}}{\lambda_1 - \lambda_2} \cdot \frac{(\lambda_1 \lambda_2)^{-(2l+1)} - 1}{1 - \lambda_1 \lambda_2} \\ = - \sum_l \frac{2l+1}{N(Z)^2} \cdot \frac{\lambda_1^{2l+1} - \lambda_2^{-(2l+1)}}{(\lambda_1 - \lambda_2)(1 - \lambda_1 \lambda_2)} - \sum_l \frac{2l+1}{N(Z)^2} \cdot \frac{\lambda_1^{-(2l+1)} - \lambda_2^{2l+1}}{(\lambda_1 - \lambda_2)(1 - \lambda_1 \lambda_2)}. \end{aligned}$$

The first sum converges absolutely if $|\lambda_1| < 1$ and $|\lambda_2| > 1$:

$$\begin{aligned} \sum_l \frac{2l+1}{N(Z)^2} \cdot \frac{\lambda_1^{2l+1} - \lambda_2^{-(2l+1)}}{(\lambda_1 - \lambda_2)(1 - \lambda_1 \lambda_2)} &= \frac{N(Z)^{-2}}{(\lambda_1 - \lambda_2)(1 - \lambda_1 \lambda_2)} \left(\frac{\lambda_1}{(1 - \lambda_1)^2} - \frac{\lambda_2}{(1 - \lambda_2)^2} \right) \\ &= \frac{N(Z)^{-2}}{(1 - \lambda_1)^2(1 - \lambda_2)^2} = \frac{N(Z)^{-2}}{N(1 - Z^{-1}W)^2} = \frac{1}{N(Z - W)^2}. \end{aligned}$$

The second sum converges absolutely if $|\lambda_1| > 1$ and $|\lambda_2| < 1$:

$$\sum_l \frac{2l+1}{N(Z)^2} \cdot \frac{\lambda_1^{-(2l+1)} - \lambda_2^{2l+1}}{(\lambda_1 - \lambda_2)(1 - \lambda_1 \lambda_2)} = \frac{1}{N(Z - W)^2}.$$

Of course, the set of Z and W where both sums converge absolutely is empty, but these formal calculations strongly suggest that there is a way to make sense of the series (32) in terms of distributions:

$$\begin{aligned} \sum_{k,l,m,n} (2l+1)t_{n\underline{m}}^l(Z^{-1}) \cdot N(Z)^{-k-2} \cdot t_{m\underline{n}}^l(W) \cdot N(W)^k \\ = - \left(\text{Reg}_+ \frac{1}{N(Z - W)^2} + \text{Reg}_- \frac{1}{N(Z - W)^2} \right) \quad (33) \end{aligned}$$

with $ZW^{-1} \in U(2)$ and $\text{Reg}_{\pm} \frac{1}{N(Z - W)^2}$ denoting some sort of regularizations of $\frac{1}{N(Z - W)^2}$.

6 Equivariant Embeddings of and Projectors onto the Irreducible Components of (ρ_1, \mathfrak{K})

In this section we construct $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariant embeddings of the irreducible components of (ρ_1, \mathfrak{K}) into tensor products $(\pi_l^0, \mathcal{H}^{\pm}) \otimes (\pi_r^0, \mathcal{H}^{\pm})$ with the property that, when composed with the multiplication map, the result is the identity map on that irreducible component. The tensor product $(\pi_l^0, \mathcal{H}^+) \otimes (\pi_r^0, \mathcal{H}^+)$ was decomposed into a direct sum of irreducible components in [JV] with (ρ_1, \mathfrak{K}^+) being one of these components, and it was shown that each irreducible component has multiplicity one. Hence the $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariant map $\mathfrak{K}^+ \rightarrow \mathcal{H}^+ \otimes \mathcal{H}^+$ is unique up to a scalar multiple. Dually, the multiplicity of (ρ_1, \mathfrak{K}^-) in $(\pi_l^0, \mathcal{H}^-) \otimes (\pi_r^0, \mathcal{H}^-)$ is one and the $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariant map $\mathfrak{K}^- \rightarrow \mathcal{H}^- \otimes \mathcal{H}^-$ is unique up to a scalar multiple as well. On the other hand, the equivariant embedding $\mathfrak{K}^0 \hookrightarrow \mathcal{H}^- \otimes \mathcal{H}^+$ requires a more subtle approach. As an immediate application of these embedding maps we obtain projectors of (ρ_1, \mathfrak{K}) onto its irreducible components.

We consider the maps

$$\mathfrak{K} \ni f \mapsto (I_R f)(Z_1, Z_2) = \frac{i}{2\pi^3} \int_{W \in U(2)_R} \frac{f(W) dV}{N(W - Z_1) \cdot N(W - Z_2)} \in \overline{\mathcal{H} \otimes \mathcal{H}}, \quad (34)$$

where $\overline{\mathcal{H} \otimes \mathcal{H}}$ denotes the Hilbert space obtained by completing $\mathcal{H} \otimes \mathcal{H}$ with respect to the unitary structure coming from the tensor product of unitary representations (π_l^0, \mathcal{H}) and (π_r^0, \mathcal{H}) . If $Z_1, Z_2 \in \mathbb{D}_R^- \sqcup \mathbb{D}_R^+$, the integrand has no singularities and the result is a holomorphic function in two variables Z_1, Z_2 which is harmonic in each variable separately. We will see soon that the result depends on whether Z_1 and Z_2 are both in \mathbb{D}_R^+ , both in \mathbb{D}_R^- or one is in \mathbb{D}_R^+ and the other is in \mathbb{D}_R^- . Thus the expression (34) determines four different maps.

Lemma 11. *The maps $f \mapsto (I_R f)(Z_1, Z_2)$ are $U(2, 2)_R$ and $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariant.*

Proof. We need to show that, for all $h \in U(2, 2)_R$, the maps (34) commute with the action of h . Writing $h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\tilde{Z}_1 = (aZ_1 + b)(cZ_1 + d)^{-1}, \quad \tilde{Z}_2 = (aZ_2 + b)(cZ_2 + d)^{-1}, \quad \tilde{W} = (aW + b)(cW + d)^{-1}$$

and using Lemmas 1 and 2 we obtain:

$$\begin{aligned} & \int_{W \in U(2)_R} \frac{(\rho_1(h)f)(W) dV}{N(W - Z_1) \cdot N(W - Z_2)} \\ &= \int_{W \in U(2)_R} \frac{f(\tilde{W}) \cdot N(cW + d)^{-2} \cdot N(a' - Wc')^{-2} dV}{N(\tilde{W} - \tilde{Z}_1) \cdot N(\tilde{W} - \tilde{Z}_2) \cdot N(cZ_1 + d) \cdot N(a' - Z_2c')} \\ &= \frac{1}{N(cZ_1 + d) \cdot N(a' - Z_2c')} \int_{\tilde{W} \in U(2)_R} \frac{f(\tilde{W}) dV}{N(\tilde{W} - \tilde{Z}_1) \cdot N(\tilde{W} - \tilde{Z}_2)}. \end{aligned}$$

This proves the $U(2, 2)_R$ -equivariance. The $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariance then follows since $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}}) \simeq \mathbb{C} \otimes \mathfrak{u}(2, 2)_R$. \square

Now we compose the embedding maps I_R with the multiplication map M defined by (29).

Theorem 12. *The maps $f \mapsto (I_R f)(Z_1, Z_2)$ have the following properties:*

1. If $Z_1, Z_2 \in \mathbb{D}_R^+$, then $I_R : \mathfrak{K} \rightarrow \mathcal{H}^+ \otimes \mathcal{H}^+$,

$$M \circ (I_R f)(Z_1, Z_2) = f \quad \text{if } f \in \mathfrak{K}^+ \quad \text{and} \quad (I_R f)(Z_1, Z_2) = 0 \quad \text{if } f \in \mathfrak{K}^- \oplus \mathfrak{K}^0;$$

2. If $Z_1, Z_2 \in \mathbb{D}_R^-$, then $I_R : \mathcal{H} \rightarrow \mathcal{H}^- \otimes \mathcal{H}^-$,

$$M \circ (I_R f)(Z_1, Z_2) = f \quad \text{if } f \in \mathcal{H}^- \quad \text{and} \quad (I_R f)(Z_1, Z_2) = 0 \quad \text{if } f \in \mathcal{H}^0 \oplus \mathcal{H}^+.$$

Proof. We prove part 1 only, the other part can be proven in the same way. Note that the representations (ρ_1, \mathcal{H}^-) , (ρ_1, \mathcal{H}^0) and (ρ_1, \mathcal{H}^+) are generated by $N(W)^{-2}$, $N(W)^{-1}$ and 1 respectively. For this reason we compute $(I_R N(W)^k)(Z_1, Z_2)$ for $k = -2, -1, 0$. Suppose $Z_1, Z_2 \in \mathbb{D}_R^+$ and use the matrix coefficient expansion given by Proposition 25 in [FL1]

$$\frac{1}{N(Z - W)} = N(W)^{-1} \cdot \sum_{l, m, n} t_{m \underline{n}}^l(Z) \cdot t_{n \underline{m}}^l(W^{-1}), \quad \begin{array}{l} l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \\ m, n = -l, -l+1, \dots, l, \end{array} \quad (35)$$

which converges pointwise absolutely in the region $\{(Z, W) \in \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}}^{\times}; ZW^{-1} \in \mathbb{D}^+\}$. We compute:

$$\begin{aligned} (I_R N(W)^k)(Z_1, Z_2) &= \frac{i}{2\pi^3} \int_{W \in U(2)_R} \frac{N(W)^k dV}{N(W - Z_1) \cdot N(W - Z_2)} \\ &= \left\langle \frac{N(W)^k}{N(W - Z_1)}, \frac{1}{N(W^+ - Z_2^+)} \right\rangle_W \\ &= \sum_{l, m, n, l', m', n'} t_{n \underline{m}}^l(Z_1) \cdot t_{m' \underline{n}'}^{l'}(Z_2^+) \cdot \langle N(W)^{k-2-2l'} \cdot t_{m \underline{n}}^l(W^{-1}), t_{n' \underline{m}'}^{l'}(W) \rangle. \end{aligned}$$

By the orthogonality relations (19) this is zero unless $l = l'$, $m = m'$, $n = n'$ and $k = 2l$. Therefore,

$$(I_R N(W)^k)(Z_1, Z_2) = \begin{cases} 0 & \text{if } k = -2, -1; \\ 1 & \text{if } k = 0. \end{cases}$$

By $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariance (Lemma 11) we see that $(I_R f)(Z_1, Z_2)$ is always a polynomial in Z_1 and Z_2 , hence an element of $\mathcal{H}^+ \otimes \mathcal{H}^+$ and part 1 follows. \square

Example 13. Using similar computations one can show that

$$(I_R N(W))(Z_1, Z_2) = \frac{1}{2} \text{Tr}(Z_1 Z_2^+) \quad \text{and} \quad (I_R w_{ij})(Z_1, Z_2) = \frac{1}{2}((z_{ij})_1 + (z_{ij})_2), \quad Z_1, Z_2 \in \mathbb{D}_R^+,$$

where w_{ij} denotes the ij -entry of the 2×2 matrix W .

Corollary 14. The maps $f \mapsto (I_R f)(Z_1, Z_2)$ followed by the multiplication map provide projectors onto the irreducible components of (ρ_1, \mathcal{H}) . More precisely,

1. If $Z \in \mathbb{D}_R^+$, then the map

$$f \mapsto (P^+ f)(Z) = \frac{i}{2\pi^3} \int_{W \in U(2)_R} \frac{f(W) dV}{N(W - Z)^2}$$

is a projector onto \mathcal{H}^+ ;

2. If $Z \in \mathbb{D}_R^-$, then the map

$$f \mapsto (P^- f)(Z) = \frac{i}{2\pi^3} \int_{W \in U(2)_R} \frac{f(W) dV}{N(W - Z)^2}$$

is a projector onto \mathcal{H}^- .

In particular, these maps P^+ and P^- provide reproducing formulas for functions in \mathcal{H}^+ and \mathcal{H}^- respectively.

(The reproducing formula for functions in \mathcal{H}^+ was obtained in [FL1], Theorem 70.)

Now we suppose $Z_1 \in \mathbb{D}_R^+$ and $Z_2 \in \mathbb{D}_R^-$, this case is much more subtle. Using the matrix coefficient expansion (35) of $N(Z - W)^{-1}$ one more time, we compute:

$$\begin{aligned} (I_R N(W)^k)(Z_1, Z_2) &= \frac{i}{2\pi^3} \int_{W \in U(2)_R} \frac{N(W)^k dV}{N(W - Z_1) \cdot N(W - Z_2)} \\ &= \left\langle \frac{N(W)^k}{N(W - Z_1)}, \frac{1}{N(W - Z_2)} \right\rangle_W \\ &= N(Z_2)^{-1} \sum_{l, m, n, l', m', n'} t_{n \underline{m}}^l(Z_1) \cdot t_{m' \underline{n}'}^{l'}(Z_2^{-1}) \cdot \langle N(W)^{k-1} \cdot t_{m \underline{n}}^l(W^{-1}), t_{n' \underline{m}'}^{l'}(W) \rangle. \end{aligned}$$

By the orthogonality relations (19) this is zero unless $l = l'$, $m = m'$, $n = n'$ and $k = -1$. By $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariance (Lemma 11) we can conclude that $(I_R f)(Z_1, Z_2) = 0$ if $f \in \mathcal{H}^- \oplus \mathcal{H}^+$. So, let us assume now $l = l'$, $m = m'$, $n = n'$ and $k = -1$. In this case we get

$$(I_R N(W)^{-1})(Z_1, Z_2) = \sum_{l, m, n} \frac{N(Z_2)^{-1}}{2l+1} t_{n \underline{m}}^l(Z_1) \cdot t_{m \underline{n}}^l(Z_2^{-1}) = \sum_{l, n} \frac{N(Z_2)^{-1}}{2l+1} t_{n \underline{n}}^l(Z_1 \cdot Z_2^{-1}).$$

Assume further that $Z_1 \cdot Z_2^{-1}$ can be diagonalized as $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1 \neq \lambda_2$. This is allowed since the set of matrices with different eigenvalues is dense in $\mathbb{H}_{\mathbb{C}}$. Since $Z_1 \in \mathbb{D}_R^+$ and $Z_2 \in \mathbb{D}_R^-$, we have $|\lambda_1|, |\lambda_2| < 1$. Recall that χ_l denotes the character of the irreducible representation of $GL(2, \mathbb{C})$ of dimension $2l+1$ and $\chi_l(Z_1 \cdot Z_2^{-1}) = \frac{\lambda_1^{2l+1} - \lambda_2^{2l+1}}{\lambda_1 - \lambda_2}$. Hence

$$\begin{aligned} (I_R N(W)^{-1})(Z_1, Z_2) &= \sum_l \frac{N(Z_2)^{-1}}{2l+1} \chi_l(Z_1 \cdot Z_2^{-1}) \\ &= \sum_l \frac{N(Z_2)^{-1}}{2l+1} \frac{\lambda_1^{2l+1} - \lambda_2^{2l+1}}{\lambda_1 - \lambda_2} = \frac{N(Z_2)^{-1}}{\lambda_2 - \lambda_1} \log \left(\frac{1 - \lambda_1}{1 - \lambda_2} \right). \quad (36) \end{aligned}$$

Although this expression is valid only in the region where $\lambda_1 \neq \lambda_2$, the right hand side clearly continues analytically across the set of $Z_1 \cdot Z_2^{-1}$ for which $\lambda_1 = \lambda_2$. However, this is obviously not a polynomial in $Z_1, Z_2, N(Z_1)^{-1}, N(Z_2)^{-1}$ and hence not an element of $\mathcal{H} \otimes \mathcal{H}$. Note that composing $(I_R N(W)^{-1})(Z_1, Z_2)$ with the multiplication map M amounts to setting $Z_1 = Z_2 = Z$ and letting $\lambda_1, \lambda_2 \rightarrow 1$, but then the limit is infinite! To get around this problem, observe that (36) remains valid if we let Z_1 and Z_2 approach two different points in $U(2)_R$ so that $Z_1 \in \mathbb{D}_R^+$ and $Z_2 \in \mathbb{D}_R^-$. Thus we have a well defined operator

$$f \mapsto (I_R^{+-} f)(Z_1, Z_2) = \frac{i}{2\pi^3} \lim_{\substack{Z'_1 \rightarrow Z_1, Z'_1 \in \mathbb{D}_R^+ \\ Z'_2 \rightarrow Z_2, Z'_2 \in \mathbb{D}_R^-}} \int_{W \in U(2)_R} \frac{f(W) dV}{N(W - Z'_1) \cdot N(W - Z'_2)},$$

where $Z_1, Z_2 \in U(2)_R$ and none of the eigenvalues of $Z_1 \cdot Z_2^{-1}$ is 1, i.e. $N(Z_1 - Z_2) \neq 0$. Similarly, we can switch the roles of Z_1 and Z_2 and define another operator

$$f \mapsto (I_R^{-+} f)(Z_1, Z_2) = \frac{i}{2\pi^3} \lim_{\substack{Z'_1 \rightarrow Z_1, Z'_1 \in \mathbb{D}_R^- \\ Z'_2 \rightarrow Z_2, Z'_2 \in \mathbb{D}_R^+}} \int_{W \in U(2)_R} \frac{f(W) dV}{N(W - Z'_1) \cdot N(W - Z'_2)},$$

where $Z_1, Z_2 \in U(2)_R$ and $N(Z_1 - Z_2) \neq 0$.

It follows from Lemma 11 that the operators I_R^{+-} and I_R^{-+} are $U(2, 2)_R$ -equivariant. We already know that these operators annihilate $N(Z)^k$ for $k \neq -1$. Hence they annihilate the entire $\mathcal{K}^- \oplus \mathcal{K}^+$. Next we compute the limit

$$\lim_{Z_1, Z_2 \rightarrow Z} ((I_R^{+-} + I_R^{-+})N(W)^{-1})(Z_1, Z_2), \quad Z \in U(2)_R.$$

As before, suppose that $Z_1 \cdot Z_2^{-1}$ has eigenvalues λ_1 and λ_2 with $|\lambda_1| = |\lambda_2| = 1$, $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$. Then $N(Z_1) = \lambda_1 \lambda_2 \cdot N(Z_2)$, $Z_2 \cdot Z_1^{-1}$ has eigenvalues λ_1^{-1} and λ_2^{-1} . Assume for a moment that $\lambda_1 \neq \lambda_2$, then by (36) we have:

$$\begin{aligned} ((I_R^{+-} + I_R^{-+})N(W)^{-1})(Z_1, Z_2) &= \frac{N(Z_2)^{-1}}{\lambda_2 - \lambda_1} \log\left(\frac{1 - \lambda_1}{1 - \lambda_2}\right) + \frac{N(Z_1)^{-1}}{\lambda_2^{-1} - \lambda_1^{-1}} \log\left(\frac{1 - \lambda_1^{-1}}{1 - \lambda_2^{-1}}\right) \\ &= \frac{N(Z_2)^{-1}}{\lambda_2 - \lambda_1} \log\left(\frac{1 - \lambda_1}{1 - \lambda_2}\right) - \frac{N(Z_2)^{-1}}{\lambda_2 - \lambda_1} \log\left(\frac{\lambda_2(\lambda_1 - 1)}{\lambda_1(\lambda_2 - 1)}\right) = -\frac{1}{N(Z_2)} \frac{\log \lambda_2 - \log \lambda_1}{\lambda_2 - \lambda_1}. \end{aligned}$$

Hence,

$$((I_R^{+-} + I_R^{-+})N(W)^{-1})(Z_1, Z_2) = -\frac{1}{N(Z_2)} \cdot \begin{cases} \frac{\log \lambda_2 - \log \lambda_1}{\lambda_2 - \lambda_1} & \text{if } \lambda_1 \neq \lambda_2; \\ \lambda^{-1} & \text{if } \lambda_1 = \lambda_2 = \lambda. \end{cases}$$

Therefore,

$$\lim_{\substack{Z_1, Z_2 \rightarrow Z \\ N(Z_1 - Z_2) \neq 0}} ((I_R^{+-} + I_R^{-+})N(W)^{-1})(Z_1, Z_2) = -N(Z)^{-1}, \quad Z \in U(2)_R.$$

From the $U(2, 2)_R$ -equivariance we see that we have obtained a projector onto \mathcal{K}^0 :

Theorem 15. *The $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$ -equivariant map*

$$f \mapsto ((I_R^{+-} + I_R^{-+})f)(Z_1, Z_2) \in \overline{\mathcal{H} \otimes \mathcal{H}}, \quad f \in \mathcal{K}, \quad Z_1, Z_2 \in U(2)_R,$$

is well-defined, annihilates $\mathcal{K}^- \oplus \mathcal{K}^+$ and satisfies

$$M \circ ((I_R^{+-} + I_R^{-+})f) = f \quad \text{if } f \in \mathcal{K}^0.$$

In particular, an operator P^0 on \mathcal{K}

$$f \mapsto (P^0 f)(Z) = -\lim_{\substack{Z_1, Z_2 \rightarrow Z \\ N(Z_1 - Z_2) \neq 0}} ((I_R^{+-} + I_R^{-+})f)(Z_1, Z_2), \quad Z \in U(2)_R,$$

is well-defined, annihilates $\mathcal{K}^- \oplus \mathcal{K}^+$ and is the identity mapping on \mathcal{K}^0 .

Finally, the operator P^0 on \mathcal{K} can be computed as follows:

$$\begin{aligned} (P^0 f)(Z) &= \frac{1}{2\pi^3 i} \lim_{\theta \rightarrow 0} \lim_{s \rightarrow 1} \left(\int_{W \in U(2)_R} \frac{f(W) dV}{N(W - se^{i\theta} Z) \cdot N(W - s^{-1}e^{-i\theta} Z)} \right. \\ &\quad \left. + \int_{W \in U(2)_R} \frac{f(W) dV}{N(W - s^{-1}e^{i\theta} Z) \cdot N(W - se^{-i\theta} Z)} \right), \quad Z \in U(2)_R. \end{aligned}$$

Note that the space \mathcal{K} consists of rational functions, and rational functions on $\mathbb{H}_\mathbb{C}$ as well as analytic ones are completely determined by their values on $U(2)_R$. Note also that this integral formula for P^0 is in complete agreement with our previous formal computation (33) of the reproducing kernel for (ρ_1, \mathcal{K}^0) .

Remark 16. *Every function $f \in \mathcal{K}$ can be written as $f = P^- f + P^0 f + P^+ f$. Combining the integral expressions for $P^\pm f$ and $P^0 f$ obtained in Corollary 14 and Theorem 15 we get a reproducing formula for all functions in \mathcal{K} that is equivalent to (9).*

7 The One-Loop Feynman Integral and Its Relation to

$$(\pi_l^0, \mathcal{H}^\pm) \otimes (\pi_r^0, \mathcal{H}^\pm)$$

In this section we show that the identification of the one-loop Feynman diagram with the integral kernel $p_1^0(Z_1, Z_2; W_1, W_2)$ of the integral operators expressing \mathcal{P}^+ and \mathcal{P}^- found in [FL1] is an immediate consequence of Theorem 12. These operators \mathcal{P}^+ and \mathcal{P}^- are the $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$ -equivariant composition maps

$$\mathcal{P}^+ : \mathcal{H}^+ \otimes \mathcal{H}^+ \rightarrow \mathcal{K}^+ \hookrightarrow \mathcal{H}^+ \otimes \mathcal{H}^+ \quad \text{and} \quad \mathcal{P}^- : \mathcal{H}^- \otimes \mathcal{H}^- \rightarrow \mathcal{K}^- \hookrightarrow \mathcal{H}^- \otimes \mathcal{H}^- \quad (37)$$

(the multiplication map followed by the embedding). As we mentioned earlier, the multiplicities of (ρ_1, \mathcal{K}^+) in $(\pi_l^0, \mathcal{H}^+) \otimes (\pi_r^0, \mathcal{H}^+)$ and of (ρ_1, \mathcal{K}^-) in $(\pi_l^0, \mathcal{H}^-) \otimes (\pi_r^0, \mathcal{H}^-)$ are both one. So the maps \mathcal{P}^+ and \mathcal{P}^- are unique up to multiplication by scalars and they are pinned down by imposing

$$\mathcal{P}^+(1 \otimes 1) = 1 \otimes 1 \quad \text{and} \quad \mathcal{P}^-(N(Z_1)^{-1} \otimes N(Z_2)^{-1}) = N(W_1)^{-1} \otimes N(W_2)^{-1}.$$

For convenience we restate Theorem 34 and Corollary 39 from [FL1]. We define operators on \mathcal{H} by

$$\begin{aligned} (S_R^+ \varphi)(Z) &= \frac{1}{2\pi^2} \int_{X \in S_R^3} \frac{(\widetilde{\deg \varphi})(X)}{N(X - Z)} \cdot \frac{dS}{R}, & Z \in \mathbb{D}_R^+, \\ (S_R^- \varphi)(Z) &= \frac{1}{2\pi^2} \int_{X \in S_R^3} \frac{(\widetilde{\deg \varphi})(X)}{N(X - Z)} \cdot \frac{dS}{R}, & Z \in \mathbb{D}_R^-. \end{aligned}$$

Theorem 17. *The operators S_R^- and S_R^+ are continuous linear operators $\mathcal{H} \rightarrow \mathcal{H}$. The operator S_R^+ has image in \mathcal{H}^+ and sends*

$$\begin{aligned} t_{n\underline{m}}^l(X) &\mapsto t_{n\underline{m}}^l(Z), & l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots, \\ & & m, n \in \mathbb{Z} + l, \\ t_{n\underline{m}}^l(X) \cdot N(X)^{-2l-1} &\mapsto -R^{-2(2l+1)} \cdot t_{n\underline{m}}^l(Z), & -l \leq m, n \leq l. \end{aligned}$$

The operator S_R^- has image in \mathcal{H}^- and sends

$$\begin{aligned} t_{n\underline{m}}^l(X) &\mapsto R^{2(2l+1)} \cdot N(Z)^{-2l-1} \cdot t_{n\underline{m}}^l(Z), & l = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots, \\ & & m, n \in \mathbb{Z} + l, \\ t_{n\underline{m}}^l(X) \cdot N(X)^{-2l-1} &\mapsto -t_{n\underline{m}}^l(Z) \cdot N(Z)^{-2l-1}, & -l \leq m, n \leq l. \end{aligned}$$

Now, let us take a close look at the function of three variables

$$\frac{1}{N(W - Z_1) \cdot N(W - Z_2)}.$$

On the one hand, this function has appeared in (34) and is responsible for $\mathfrak{gl}(2, \mathbb{H}_\mathbb{C})$ -equivariant embeddings of \mathcal{K}^\pm into $\mathcal{H}^\pm \otimes \mathcal{H}^\pm$. On the other hand, as can be seen from Theorem 17, this function can be used to express the multiplication maps (29):

Lemma 18. *Fix $R_1, R_2 > 0$ and consider a map \widetilde{M} on $\mathcal{H} \otimes \mathcal{H}$ sending pure tensors*

$$\varphi_1(Z_1) \otimes \varphi_2(Z_2) \mapsto \frac{1}{(2\pi^2)^2} \int \int_{\substack{Z_1 \in S_{R_1}^3 \\ Z_2 \in S_{R_2}^3}} \frac{(\widetilde{\deg \varphi_1})(Z_1) \cdot (\widetilde{\deg \varphi_2})(Z_2)}{N(W - Z_1) \cdot N(W - Z_2)} \cdot \frac{dS_1 dS_2}{R_1 R_2}.$$

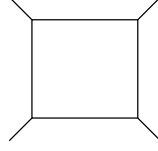


Figure 4: One-loop Feynman diagram

1. If $\varphi_1, \varphi_2 \in \mathcal{H}^+$ and $W \in \mathbb{D}_{R_1}^+ \cap \mathbb{D}_{R_2}^+$, then \widetilde{M} is the multiplication map:

$$\widetilde{M}(\varphi_1(Z_1) \otimes \varphi_2(Z_2)) = (\varphi_1 \cdot \varphi_2)(W);$$

2. If $\varphi_1, \varphi_2 \in \mathcal{H}^-$ and $W \in \mathbb{D}_{R_1}^- \cap \mathbb{D}_{R_2}^-$, then \widetilde{M} is the multiplication map:

$$\widetilde{M}(\varphi_1(Z_1) \otimes \varphi_2(Z_2)) = (\varphi_1 \cdot \varphi_2)(W);$$

3. If $\varphi_1 \in \mathcal{H}^+$, $\varphi_2 \in \mathcal{H}^-$ and $W \in \mathbb{D}_{R_1}^+ \cap \mathbb{D}_{R_2}^-$, then \widetilde{M} is the negative of the multiplication map:

$$\widetilde{M}(\varphi_1(Z_1) \otimes \varphi_2(Z_2)) = -(\varphi_1 \cdot \varphi_2)(W).$$

Combining Theorem 12 and Lemma 18 we see that the function

$$p_1^0(Z_1, Z_2; W_1, W_2) = \frac{i}{2\pi^3} \int_{T \in U(2)} \frac{dV}{N(Z_1 - T) \cdot N(Z_2 - T) \cdot N(W_1 - T) \cdot N(W_2 - T)}$$

can be interpreted as the integral kernel of the integral operators expressing the $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariant compositions (37). Explicitly, we have:

$$\begin{aligned} & (\mathcal{P}^+(\varphi_1 \otimes \varphi_2))(W_1, W_2) \\ &= \frac{1}{(2\pi^2)^2} \iint_{\substack{Z_1 \in S_{R_1}^3 \\ Z_2 \in S_{R_2}^3}} p_1^0(Z_1, Z_2; W_1, W_2) \cdot (\widetilde{\deg}_{Z_1} \varphi_1)(Z_1) \cdot (\widetilde{\deg}_{Z_2} \varphi_2)(Z_2) \frac{dS_1 dS_2}{R_1 R_2}, \end{aligned}$$

where $\varphi_1, \varphi_2 \in \mathcal{H}^+$, $W_1, W_2 \in \mathbb{D}_1^+$ and $R_1, R_2 > 1$. Similarly,

$$\begin{aligned} & (\mathcal{P}^-(\varphi_1 \otimes \varphi_2))(W_1, W_2) \\ &= \frac{1}{(2\pi^2)^2} \iint_{\substack{Z_1 \in S_{R_1}^3 \\ Z_2 \in S_{R_2}^3}} p_1^0(Z_1, Z_2; W_1, W_2) \cdot (\widetilde{\deg}_{Z_1} \varphi_1)(Z_1) \cdot (\widetilde{\deg}_{Z_2} \varphi_2)(Z_2) \frac{dS_1 dS_2}{R_1 R_2}, \end{aligned}$$

where $\varphi_1, \varphi_2 \in \mathcal{H}^-$, $W_1, W_2 \in \mathbb{D}_1^-$ and $0 < R_1, R_2 < 1$.

We conclude this section with a comment that the integral kernel $p_1^0(Z_1, Z_2; W_1, W_2)$ can be rewritten as an integral over \mathbb{R}^4 instead of $U(2)$, as was done after Corollary 90 in [FL1]. Thus $p_1^0(Z_1, Z_2; W_1, W_2)$ gets identified with the integral represented by the one-loop Feynman diagram (see Figure 4).

8 Minkowski Space Realization of \mathfrak{K}^- , \mathfrak{K}^0 and \mathfrak{K}^+

In this section we realize the spaces \mathfrak{K}^- , \mathfrak{K}^0 and \mathfrak{K}^+ in the setting of the Minkowski space \mathbb{M} . As in [FL1], we use e_0, e_1, e_2, e_3 in place of the more familiar generators $1, i, j, k$ of \mathbb{H} , so that the symbol i can be used for $\sqrt{-1} \in \mathbb{C}$; and let $\tilde{e}_0 = -ie_0 \in \mathbb{H}_{\mathbb{C}}$. Then

$$\mathbb{M} = \tilde{e}_0 \mathbb{R} \oplus e_1 \mathbb{R} \oplus e_2 \mathbb{R} \oplus e_3 \mathbb{R} = \left\{ Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{H}_{\mathbb{C}}; z_{11}, z_{22} \in i\mathbb{R}, z_{21} = -\overline{z_{12}} \right\}.$$

Recall the generalized upper and lower half-planes introduced in Section 3.5 in [FL1]:

$$\begin{aligned}\mathbb{T}^- &= \{Z = W_1 + iW_2 \in \mathbb{H}_{\mathbb{C}}; W_1, W_2 \in \mathbb{M}, iW_2 \text{ is positive definite}\}, \\ \mathbb{T}^+ &= \{Z = W_1 + iW_2 \in \mathbb{H}_{\mathbb{C}}; W_1, W_2 \in \mathbb{M}, iW_2 \text{ is negative definite}\}\end{aligned}$$

and element $\gamma \in GL(2, \mathbb{C})$ from Lemmas 54 and 63 of [FL1] which induces a fractional linear transformation on $\mathbb{H}_{\mathbb{C}}$ that we call the ‘‘Cayley transform’’. Thus $\gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & -1 \end{pmatrix} \in GL(2, \mathbb{H}_{\mathbb{C}})$ with $\gamma^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix}$. The fractional linear map on $\mathbb{H}_{\mathbb{C}}$

$$\pi_l(\gamma) : Z \mapsto (Z - i)(Z + i)^{-1}$$

maps $\mathbb{D}^+ \rightarrow \mathbb{T}^+$, $\mathbb{D}^- \rightarrow \mathbb{T}^-$, $U(2) \rightarrow \mathbb{M}$ (with singularities) and sends the sphere $\{Z \in U(2); N(Z) = 1\} = SU(2)$ into the two-sheeted hyperboloid $\{Y \in \mathbb{M}; N(Y) = -1\}$. Conversely, the fractional linear map on $\mathbb{H}_{\mathbb{C}}$

$$\pi_l(\gamma^{-1}) : Z \mapsto -i(Z + 1)(Z - 1)^{-1}$$

maps $\mathbb{T}^+ \rightarrow \mathbb{D}^+$, $\mathbb{T}^- \rightarrow \mathbb{D}^-$, $\mathbb{M} \rightarrow U(2)$, has no singularities on \mathbb{M} , and sends the two-sheeted hyperboloid $\{Y \in \mathbb{M}; N(Y) = -1\}$ into the sphere $\{Z \in U(2); N(Z) = 1\} = SU(2)$.

These fractional linear transformations induce the following maps on functions:

$$\pi_l^0(\gamma) : \varphi(Z) \mapsto (\pi_l^0(\gamma)\varphi)(Z) = \frac{2}{N(Z - 1)} \cdot \varphi(-i(Z + 1)(Z - 1)^{-1}),$$

sends harmonic functions² on \mathbb{D}^+ , \mathbb{D}^- and $U(2)$ into solutions of the wave equation on, respectively, \mathbb{T}^+ , \mathbb{T}^- and \mathbb{M} . Similarly,

$$\pi_l^0(\gamma^{-1}) : \varphi(Z) \mapsto (\pi_l^0(\gamma^{-1})\varphi)(Z) = \frac{-2}{N(Z + i)} \cdot \varphi((Z - i)(Z + i)^{-1}),$$

sends solutions of the wave equation on \mathbb{T}^+ , \mathbb{T}^- and \mathbb{M} into harmonic functions on, respectively, \mathbb{D}^+ , \mathbb{D}^- and $U(2)$. In particular, $\pi_l^0(\gamma)$ maps

$$1 \mapsto 2 \cdot N(Z - 1)^{-1}, \quad N(Z)^{-1} \mapsto -2 \cdot N(Z + 1)^{-1}. \quad (38)$$

The light cone

$$\text{Cone} = \{Y \in \mathbb{M}; N(Y) = 0\}$$

can be divided into two parts:

$$\text{Cone}^+ = \{Y \in \text{Cone}; i \text{Tr } Y \geq 0\} \quad \text{and} \quad \text{Cone}^- = \{Y \in \text{Cone}; i \text{Tr } Y \leq 0\}.$$

Next we calculate the (inverse) Fourier transform of the delta distributions on Cone^+ and Cone^- :

Lemma 19. *We have the following absolutely convergent expansions:*

$$\begin{aligned}\frac{1}{N(Z)} &= \frac{1}{4\pi} \int_{P \in \text{Cone}^-} e^{i\langle Z, P \rangle} \frac{dp^1 dp^2 dp^3}{|p^0|}, & Z \in \mathbb{T}^-, \\ \frac{1}{N(Z)} &= \frac{1}{4\pi} \int_{P \in \text{Cone}^+} e^{i\langle Z, P \rangle} \frac{dp^1 dp^2 dp^3}{|p^0|}, & Z \in \mathbb{T}^+, \end{aligned}$$

where $\langle Z, P \rangle = \text{Tr}(Z^+ P)/2 = \text{Tr}(P^+ Z)/2$ and $P = p^0 \tilde{e}_0 + p^1 e_1 + p^2 e_2 + p^3 e_3 \in \text{Cone}^\pm \subset \mathbb{M}$.

² By harmonic functions on $U(2)$ we mean functions that are holomorphic and harmonic in some open neighborhood of $U(2)$.

Proof. Note that $iW \in i\mathbb{M}$ is positive definite if and only if $N(W) < 0$ and $i\text{Tr } W > 0$ or, equivalently, if and only if $W = w^0\tilde{e}_0 + w^1e_1 + w^2e_2 + w^3e_3 \in \mathbb{M}$ and $w^0 > |\vec{w}| = \sqrt{(w^1)^2 + (w^2)^2 + (w^3)^2}$. Similarly, $iW \in i\mathbb{M}$ is negative definite if and only if $W = w^0\tilde{e}_0 + w^1e_1 + w^2e_2 + w^3e_3 \in \mathbb{M}$ and $w^0 < |\vec{w}|$. This implies that the two integrals converge absolutely on the respective regions.

Each integral defines a complex analytic function of Z on \mathbb{T}^- and \mathbb{T}^+ respectively. Hence to establish that the integrals are equal to $N(Z)^{-1}$, it is sufficient to prove that for Z of the form $z^0\tilde{e}_0 + z^1e_1 + z^2e_2 + z^3e_3 \in \mathbb{H}_{\mathbb{C}}$ with $z^0 \in \mathbb{C}^\times$ and $\vec{z} = (z^1, z^2, z^3) \in \mathbb{R}^3$. Rotating Z if necessary, without loss of generality we can assume that $z^2 = z^3 = 0$. For $\vec{p} = (p^1, p^2, p^3) \in \mathbb{R}^3$, let

$$\begin{aligned} P_+ &= |\vec{p}|\tilde{e}_0 + p^1e_1 + p^2e_2 + p^3e_3 \in \text{Cone}^+, & |\vec{p}| &= \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2}, \\ P_- &= -|\vec{p}|\tilde{e}_0 + p^1e_1 + p^2e_2 + p^3e_3 \in \text{Cone}^-. \end{aligned}$$

Let $s = \sqrt{(p^2)^2 + (p^3)^2}$ and substitute $u = \sqrt{(p^1)^2 + s^2}$, then the integrals in question become

$$\begin{aligned} \int_{P \in \text{Cone}^\mp} e^{i\langle Z, P \rangle} \frac{dp^1 dp^2 dp^3}{|p^0|} &= \int_{\vec{p} \in \mathbb{R}^3} \exp(i(\pm z^0 |\vec{p}| + z^1 p^1)) \frac{dp^1 dp^2 dp^3}{|\vec{p}|} \\ &= 2\pi \iint_{\substack{s \geq 0 \\ -\infty < p^1 < \infty}} s \exp(i(\pm z^0 \sqrt{(p^1)^2 + s^2} + z^1 p^1)) \frac{dp^1 ds}{\sqrt{(p^1)^2 + s^2}} \\ &= 2\pi \int_{-\infty}^{\infty} \left(\int_{u \geq |p^1|} e^{i(\pm z^0 u + z^1 p^1)} du \right) dp^1 \\ &= \pm \frac{2\pi i}{z^0} \int_{-\infty}^{\infty} e^{i(\pm z^0 |p^1| + z^1 p^1)} dp^1 = \frac{4\pi}{(z^1)^2 - (z^0)^2} = \frac{4\pi}{N(Z)}. \end{aligned}$$

□

Therefore,

$$\begin{aligned} \frac{1}{N(Z_1 - Z_2)} &= \frac{1}{4\pi} \int_{P \in \text{Cone}^-} e^{i\langle Z_1 - Z_2, P \rangle} \frac{dp^1 dp^2 dp^3}{|p^0|}, & \text{whenever } Z_1 - Z_2 \in \mathbb{T}^-, \\ \text{and } \frac{1}{N(Z_1 - Z_2)} &= \frac{1}{4\pi} \int_{P \in \text{Cone}^+} e^{i\langle Z_1 - Z_2, P \rangle} \frac{dp^1 dp^2 dp^3}{|p^0|}, & \text{whenever } Z_1 - Z_2 \in \mathbb{T}^+. \end{aligned}$$

Corollary 20. *Up to proportionality coefficients, the Fourier transforms of the following distributions on \mathbb{M} are:*

$$\begin{aligned} \text{the FT of } \frac{1}{N(Y-1)} & \text{ is the distribution } f \mapsto \int_{P \in \text{Cone}^+} f(P) \cdot e^{-i\text{Tr } P/2} \frac{dp^1 dp^2 dp^3}{|p^0|}, \\ \text{the FT of } \frac{1}{N(Y+1)} & \text{ is the distribution } f \mapsto \int_{P \in \text{Cone}^-} f(P) \cdot e^{i\text{Tr } P/2} \frac{dp^1 dp^2 dp^3}{|p^0|}. \end{aligned}$$

(The presence of the rapidly decaying term $e^{\pm i\text{Tr } P/2}$ ensures convergence of the integrals.)

Combining this corollary with (38), we see that the Fourier transform maps $\pi_l^0(\gamma)(\mathcal{H}^+)$ into distributions supported on Cone^+ and $\pi_l^0(\gamma)(\mathcal{H}^-)$ into distributions supported on Cone^- . Since the Fourier transform maps products of functions into convolutions, by Lemma 8 the Fourier transform maps $\rho_1(\mathcal{H}^+)$, $\rho_1(\mathcal{H}^-)$ and $\rho_1(\mathcal{H}^0)$ into distributions supported respectively in $\{Y \in \mathbb{M}; N(Y) < 0, i\text{Tr } Y \geq 0\}$ – the “interior of Cone^+ ”, $\{Y \in \mathbb{M}; N(Y) < 0, i\text{Tr } Y \leq 0\}$ – the “interior of Cone^- ” and $\{Y \in \mathbb{M}; N(Y) > 0\}$ – the “exterior of Cone ”.

Next we set $R = 1$ and pull back the maps I_1 defined by (34) via $\pi_l(\gamma^{-1})$. Using Lemmas 1 and 2 we obtain a formula that formally looks like (34):

$$\mathcal{K} \ni f \mapsto (I_1 f)(Z_1, Z_2) = \frac{i}{2\pi^3} \int_{Y \in \mathbb{M}} \frac{f(Y) dV}{N(Y - Z_1) \cdot N(Y - Z_2)} \in \overline{\mathcal{H} \otimes \mathcal{H}};$$

however, the integration is over $Y \in \mathbb{M}$, the two copies of \mathcal{H} are realized as solutions of the wave equation on \mathbb{M} and $Z_1, Z_2 \in \mathbb{T}^- \sqcup \mathbb{T}^+$. Setting $Z_1 = Z_2 \in \mathbb{T}^-$ and $Z_1 = Z_2 \in \mathbb{T}^+$ results in projectors of $\rho_1(\mathcal{K})$ onto $\rho_1(\mathcal{K}^-)$ and $\rho_1(\mathcal{K}^+)$ respectively.

Theorem 21. *Let $f \in \rho_1(\mathcal{K})$. If $Z \in \mathbb{T}^+$, then the map*

$$f \mapsto P_{\mathbb{M}}^+(Z) = \frac{i}{2\pi^3} \int_{Y \in \mathbb{M}} \frac{f(Y) dV}{N(Y - Z)^2}$$

is a projector onto $\rho_1(\mathcal{K}^+)$ and, in particular, provides a reproducing formula for functions in $\rho_1(\mathcal{K}^+)$. Similarly, if $Z \in \mathbb{T}^-$, then the map

$$f \mapsto P_{\mathbb{M}}^-(Z) = \frac{i}{2\pi^3} \int_{Y \in \mathbb{M}} \frac{f(Y) dV}{N(Y - Z)^2}$$

is a projector onto $\rho_1(\mathcal{K}^-)$ and, in particular, provides a reproducing formula for functions in $\rho_1(\mathcal{K}^-)$.

(The reproducing formulas for $\rho_1(\mathcal{K}^+)$ and $\rho_1(\mathcal{K}^-)$ were obtained in [FL1], Theorem 74.)

Next we introduce operators

$$f \mapsto (I_{\mathbb{M}}^{+-} f)(Z_1, Z_2) = \frac{i}{2\pi^3} \lim_{\substack{Z'_1 \rightarrow Z_1, Z'_1 \in \mathbb{T}^+ \\ Z'_2 \rightarrow Z_2, Z'_2 \in \mathbb{T}^-}} \int_{Y \in \mathbb{M}} \frac{f(Y) dV}{N(Y - Z'_1) \cdot N(Y - Z'_2)},$$

where $Z_1, Z_2 \in \mathbb{M}$ and $N(Z_1 - Z_2) \neq 0$. Similarly, we can switch the roles of Z_1 and Z_2 and define another operator

$$f \mapsto (I_{\mathbb{M}}^{-+} f)(Z_1, Z_2) = \frac{i}{2\pi^3} \lim_{\substack{Z'_1 \rightarrow Z_1, Z'_1 \in \mathbb{T}^- \\ Z'_2 \rightarrow Z_2, Z'_2 \in \mathbb{T}^+}} \int_{Y \in \mathbb{M}} \frac{f(Y) dV}{N(Y - Z'_1) \cdot N(Y - Z'_2)},$$

where $Z_1, Z_2 \in \mathbb{M}$ and $N(Z_1 - Z_2) \neq 0$. From Theorem 15 we obtain the following result:

Theorem 22. *The $\mathfrak{gl}(2, \mathbb{H}_{\mathbb{C}})$ -equivariant map*

$$f \mapsto ((I_{\mathbb{M}}^{+-} + I_{\mathbb{M}}^{-+})f)(Z_1, Z_2) \in \overline{\mathcal{H} \otimes \mathcal{H}}, \quad f \in \rho_1(\mathcal{K}), \quad Z_1, Z_2 \in \mathbb{M},$$

is well-defined, annihilates $\rho_1(\mathcal{K}^-) \oplus \rho_1(\mathcal{K}^+)$ and satisfies

$$M \circ ((I_{\mathbb{M}}^{+-} + I_{\mathbb{M}}^{-+})f) = f \quad \text{if } f \in \rho_1(\mathcal{K}^0).$$

In particular, an operator P^0 on $\rho_1(\mathcal{K})$

$$f \mapsto (P^0 f)(Z) = - \lim_{\substack{Z_1, Z_2 \rightarrow Z \\ N(Z_1 - Z_2) \neq 0}} ((I_{\mathbb{M}}^{+-} + I_{\mathbb{M}}^{-+})f)(Z_1, Z_2), \quad Z \in \mathbb{M},$$

is well-defined, annihilates $\rho_1(\mathcal{K}^-) \oplus \rho_1(\mathcal{K}^+)$ and is the identity mapping on $\rho_1(\mathcal{K}^0)$.

Finally, the operator P^0 on $\rho_1(\mathcal{K})$ can be computed as follows:

$$(P^0 f)(Z) = \frac{1}{2\pi^3 i} \lim_{t \rightarrow 0} \lim_{s \rightarrow 0} \left(\int_{Y \in \mathbb{M}} \frac{f(Y) dV}{N(Y - Z + it + s) \cdot N(Y - Z - it - s)} + \int_{Y \in \mathbb{M}} \frac{f(Y) dV}{N(Y - Z + it - s) \cdot N(Y - Z - it + s)} \right), \quad Z \in \mathbb{M}.$$

9 Anti de Sitter Space

We consider a 5-dimensional space $\mathbb{R}^{1,4}$ with coordinates $(w^0, w^1, w^2, w^3, w^4)$ and metric coming from an indefinite inner product

$$\langle W, W' \rangle_{1,4} = w^0 w'^0 - w^1 w'^1 - w^2 w'^2 - w^3 w'^3 - w^4 w'^4.$$

Corresponding to this metric, we have a wave operator

$$\square_{1,4} = \frac{\partial^2}{(\partial w^0)^2} - \frac{\partial^2}{(\partial w^1)^2} - \frac{\partial^2}{(\partial w^2)^2} - \frac{\partial^2}{(\partial w^3)^2} - \frac{\partial^2}{(\partial w^4)^2}.$$

We introduce notations

$$\mathbb{R}_+^{1,4} = \left\{ (w^0, w^1, w^2, w^3, w^4) \in \mathbb{R}^{1,4}; w^0 > \sqrt{(w^1)^2 + (w^2)^2 + (w^3)^2 + (w^4)^2} \right\},$$

$$\|W\|_{1,4} = \sqrt{(w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2 - (w^4)^2}, \quad W \in \mathbb{R}_+^{1,4}.$$

We fix a parameter $\mu > 0$ and introduce new coordinates $(\rho, v^1, v^2, v^3, v^4)$ on $\mathbb{R}_+^{1,4}$ as follows:

$$\begin{cases} \rho = \|W\|_{1,4}, \\ v^i = (\mu\rho)^{-1} w^i, \quad i = 1, 2, 3, 4. \end{cases}$$

Then

$$\begin{cases} w^0 = \mu\rho(\mu^{-2} + (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2)^{1/2}, \\ w^i = \mu\rho v^i, \end{cases} \quad i = 1, 2, 3, 4.$$

For each $\rho > 0$, let us denote by H_ρ the single sheet of a two-sheeted hyperboloid

$$H_\rho = \{W \in \mathbb{R}^{1,4}; (w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2 - (w^4)^2 = \rho^2, w^0 > 0\}. \quad (39)$$

Let us introduce differential operators

$$\begin{aligned} \square &= \frac{\partial^2}{(\partial v^1)^2} + \frac{\partial^2}{(\partial v^2)^2} + \frac{\partial^2}{(\partial v^3)^2} + \frac{\partial^2}{(\partial v^4)^2}, \\ \deg &= v^1 \frac{\partial}{\partial v^1} + v^2 \frac{\partial}{\partial v^2} + v^3 \frac{\partial}{\partial v^3} + v^4 \frac{\partial}{\partial v^4}, \quad \widetilde{\deg} f = \deg f + f. \end{aligned}$$

By direct computation we obtain:

Lemma 23. *We have*

$$\square_{1,4} = \frac{\partial^2}{\partial \rho^2} + \frac{4}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\mu^2 \rho^2} \square_\mu, \quad (40)$$

where

$$\square_\mu = \square + \mu^2(\deg^2 + 3\deg) = \square + \mu^2(\widetilde{\deg}^2 + \widetilde{\deg} - 2).$$

We think of $\frac{\partial^2}{\partial \rho^2} + \frac{4}{\rho} \frac{\partial}{\partial \rho}$ as the “radial” part of the wave operator $\square_{1,4}$ and \square_μ as the part “tangential” to the hyperboloids H_ρ . Notice that when $\mu \rightarrow 0$, \square_μ becomes the ordinary Laplacian. We identify the space of quaternions \mathbb{H} with one sheet of a two-sheeted hyperboloid in $\mathbb{R}^{1,4}$ as follows:

$$\mathbb{H} \ni X = x^0 + ix^1 + jx^2 + kx^3 \quad \longleftrightarrow \quad (\rho, v^1 = x^0, v^2 = x^1, v^3 = x^2, v^4 = x^3) \in \mathbb{R}_+^{1,4},$$

where ρ can be any fixed positive number. We study functions on \mathbb{H} that are annihilated by the conformal Laplacian

$$\widetilde{\square}_\mu = \square + \mu^2(\deg^2 + 3\deg + 2) = \square + \mu^2(\widetilde{\deg}^2 + \widetilde{\deg}).$$

The following lemma is verified by direct computation.

Lemma 24. *Let $X, Y \in \mathbb{H}$, with Y fixed, and let*

$$\hat{X} = (\sqrt{\mu^{-2} + N(X)}, x^0, x^1, x^2, x^3) \quad \text{and} \quad \hat{Y} = (\sqrt{\mu^{-2} + N(Y)}, y^0, y^1, y^2, y^3) \in \mathbb{R}_+^{1,4},$$

so that $\mu\rho\hat{X}, \mu\rho\hat{Y} \in H_\rho$. Then

$$\widetilde{\deg}\left(\frac{1}{\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4}}\right) = \left(1 - \frac{\sqrt{\mu^{-2} + N(Y)}}{\sqrt{\mu^{-2} + N(X)}}\right) \frac{2\mu^{-2}}{(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^2}$$

and

$$\widetilde{\square}_\mu\left(\frac{1}{\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4}}\right) = 0.$$

We conclude this section with the following result.

Lemma 25. *Whenever $X, Y \in \mathbb{H}$, $X \neq Y$, we have $\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4} < 0$.*

Proof. We use two inequalities:

$$\sqrt{\mu^{-2} + N(X)}\sqrt{\mu^{-2} + N(Y)} \geq \mu^{-2} + \sqrt{N(X)N(Y)}$$

and

$$\sqrt{N(X)N(Y)} \geq x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3.$$

The first inequality is strict unless $N(X) = N(Y)$; and the second inequality is also strict unless X and Y are proportional with a non-negative proportionality coefficient. We have:

$$\begin{aligned} \langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4} &= \langle \hat{X}, \hat{X} \rangle_{1,4} + \langle \hat{Y}, \hat{Y} \rangle_{1,4} - 2\langle \hat{X}, \hat{Y} \rangle_{1,4} \\ &= 2\mu^{-2} - 2\sqrt{\mu^{-2} + N(X)}\sqrt{\mu^{-2} + N(Y)} + 2(x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3) \\ &\leq 2(x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3) - 2\sqrt{N(X)N(Y)} \leq 0, \end{aligned}$$

and if $X \neq Y$ at least one of the inequalities is strict. \square

10 Conformal Lie Algebra Action

Let $SO^+(1,4)$ denote the connected component of the identity element in $SO(1,4)$. In this section we describe the action of $SO^+(1,4)$ and its Lie algebra $\mathfrak{so}(1,4)$ on the space of solutions of $\widetilde{\square}_\mu\varphi = 0$. Then we extend the Lie algebra action to $\mathfrak{so}(1,5)$ (recall that the conformal Lie algebra in classical case is $\mathfrak{sl}(2, \mathbb{H}) \simeq \mathfrak{so}(1,5)$). Complexifying, we immediately obtain an action of $\mathbb{C} \otimes \mathfrak{so}(1,5) \simeq \mathfrak{so}(6, \mathbb{C})$. The construction of the action of $\mathfrak{so}(1,5)$ will be very similar to that of the indefinite orthogonal group $O(p,q)$ acting on the solutions of the ultrahyperbolic wave equation in $\mathbb{R}^{p-1,q-1}$. See [KØ] (and references therein) for a description of this action of $O(p,q)$ suitable for our purposes.

Fix a $\rho_0 > 0$ and recall that H_{ρ_0} denotes the single sheet of a two-sheeted hyperboloid (39). The group $SO^+(1,4)$ acts linearly on $\mathbb{R}^{1,4}$ and preserves each H_{ρ_0} . Hence it acts on functions on H_{ρ_0} by

$$\pi(a): f(W) \mapsto (\pi(a)f)(W) = f(a^{-1} \cdot W), \quad a \in SO^+(1,4). \quad (41)$$

Proposition 26. *This action preserves the kernel of $\widetilde{\square}_\mu$. That is, if φ is a function on H_{ρ_0} satisfying $\widetilde{\square}_\mu\varphi = 0$ and $a \in SO^+(1,4)$, then $\widetilde{\square}_\mu(\pi(a)\varphi) = 0$.*

Proof. Let $a \in SO^+(1, 4)$. The action of a on $\mathbb{R}^{1,4}$ commutes with the wave operator $\square_{1,4}$. Hence $\pi(a)$ commutes with the tangential part of the wave operator \square_μ . Therefore, $\pi(a)$ commutes with $\tilde{\square}_\mu = \square_\mu + 2\mu^2$. \square

In order to extend the action to $\mathfrak{so}(1, 5)$ we consider a 6-dimensional space $\mathbb{R}^{1,5}$ with coordinates $(w^0, w^1, w^2, w^3, w^4, w^5)$ and indefinite inner product

$$\langle W, W' \rangle_{1,5} = w^0 w'^0 - w^1 w'^1 - w^2 w'^2 - w^3 w'^3 - w^4 w'^4 - w^5 w'^5.$$

The group $SO(1, 5)$ acts linearly on $\mathbb{R}^{1,5}$ preserving this inner product. We introduce a function ν on $\mathbb{R}^{1,5}$:

$$\nu(W) = w^5, \quad W = (w^0, w^1, w^2, w^3, w^4, w^5) \in \mathbb{R}^{1,5}.$$

We realize $SO(1, 4)$ as the subgroup of $SO(1, 5)$ fixing the last coordinate. We can embed $\mathbb{R}^{1,4}$ into $\mathbb{R}^{1,5}$ as a hyperplane $w^5 = \text{const}$ so that $SO(1, 4)$ preserves it; and we choose to fix a particular embedding

$$\mathbb{R}^{1,4} \in (w^0, w^1, w^2, w^3, w^4) \quad \longleftrightarrow \quad (w^0, w^1, w^2, w^3, w^4, \rho_0) \in \mathbb{R}^{1,5}.$$

This way the hyperboloid H_{ρ_0} maps into the light cone in $\mathbb{R}^{1,5}$

$$\text{Cone}_{1,5} = \{W \in \mathbb{R}^{1,5}; (w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2 - (w^4)^2 - (w^5)^2 = 0\},$$

and this cone is obviously preserved by the $SO(1, 5)$ action. Let \tilde{H}_{ρ_0} be the two-sheeted hyperboloid

$$\tilde{H}_{\rho_0} = \{W \in \text{Cone}_{1,5}; w^5 = \rho_0\} \subset \text{Cone}_{1,5} \subset \mathbb{R}^{1,5},$$

then H_{ρ_0} can be identified with $\{W \in \tilde{H}_{\rho_0}; w^0 > 0\}$. The group $SO(1, 5)$ acts on \tilde{H}_{ρ_0} by projective transformations:

$$\pi(a) : \quad W \mapsto \rho_0 \frac{a \cdot W}{\nu(a \cdot W)}, \quad a \in SO(1, 5).$$

Of course, this action is defined only when $\nu(a \cdot W) \neq 0$. Then we can extend this action to functions on \tilde{H}_{ρ_0} by fixing a $\lambda \in \mathbb{C}$ and letting

$$\varpi_\lambda(a) : \quad f(W) \mapsto (\varpi_\lambda(a)f)(W) = \rho_0^{-\lambda} \cdot (\nu(a^{-1} \cdot W))^\lambda \cdot f(\pi(a^{-1})W), \quad a \in SO(1, 5).$$

Finally, we set $\lambda = -1$ and let $\pi(a) = \varpi_{-1}(a)$:

$$\pi(a) : \quad f(W) \mapsto (\pi(a)f)(W) = \frac{\rho_0}{\nu(a^{-1} \cdot W)} \cdot f(\pi(a^{-1})W), \quad a \in SO(1, 5).$$

This action extends previously defined action (41) of $SO^+(1, 4)$. Differentiating, we obtain an action of the Lie algebra $\mathfrak{so}(1, 5)$ on functions on \tilde{H}_{ρ_0} and H_{ρ_0} , which we still denote by π . Complexifying, we immediately obtain an action of $\mathbb{C} \otimes \mathfrak{so}(1, 5) \simeq \mathfrak{so}(6, \mathbb{C})$.

Theorem 27. *The π -action of the Lie algebra $\mathfrak{so}(6, \mathbb{C})$ preserves the kernel of $\tilde{\square}_\mu$. That is, if φ is a function on H_{ρ_0} satisfying $\tilde{\square}_\mu \varphi = 0$ and $h \in \mathfrak{so}(6, \mathbb{C})$, then $\tilde{\square}_\mu(\pi(h)\varphi) = 0$.*

Proof. It is sufficient to prove the result for $h \in \mathfrak{so}(1, 5) \subset \mathfrak{so}(6, \mathbb{C})$ only. For $h \in \mathfrak{so}(1, 4) \subset \mathfrak{so}(1, 5)$ the result is true by Proposition 26. As a Lie algebra, $\mathfrak{so}(1, 5)$ is generated by $\mathfrak{so}(1, 4)$ and the Lie algebra of the one-parameter family of hyperbolic rotations in the $(w^0 w^5)$ -plane:

$$a_t : \quad w^0 \mapsto w^0 \cosh t + w^5 \sinh t, \quad w^5 \mapsto w^5 \cosh t + w^0 \sinh t, \quad t \in \mathbb{R},$$

w^1, w^2, w^3 and w^4 stay unchanged. To compute $\frac{d}{dt}\big|_{t=0}\pi(a_t)$, we let $t \rightarrow 0$ and working modulo terms of order t^2 we get

$$(\pi(a_t)\varphi)(W) = \frac{\rho_0}{\rho_0 - tw^0} \cdot \varphi\left(\rho_0 \frac{w^0 - t\rho_0}{\rho_0 - tw^0}, \frac{\rho_0 w^1}{\rho_0 - tw^0}, \frac{\rho_0 w^2}{\rho_0 - tw^0}, \frac{\rho_0 w^3}{\rho_0 - tw^0}, \frac{\rho_0 w^4}{\rho_0 - tw^0}\right).$$

Rewriting it in $(\rho, v^1, v^2, v^3, v^4)$ coordinates, we obtain

$$(\pi(a_t)\varphi)(W) = \frac{\rho_0}{\rho_0 - tw^0} \cdot \varphi\left(\rho_0, \frac{\rho_0 v^1}{\rho_0 - tw^0}, \frac{\rho_0 v^2}{\rho_0 - tw^0}, \frac{\rho_0 v^3}{\rho_0 - tw^0}, \frac{\rho_0 v^4}{\rho_0 - tw^0}\right).$$

Hence

$$\frac{d}{dt}(\pi(a_t)\varphi)\big|_{t=0} = \frac{w^0}{\rho_0}\varphi + \frac{w^0}{\rho_0}\deg\varphi = \frac{w^0}{\rho_0}\widetilde{\deg}\varphi = (1 + \mu^2 N(X))^{1/2}\widetilde{\deg}\varphi,$$

since $w^0 = \rho_0(1 + \mu^2 N(X))^{1/2}$, where $N(X) = (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2$. Finally, the theorem follows from the lemma below. \square

Lemma 28. *The operator $\varphi \mapsto (1 + \mu^2 N(X))^{1/2}\widetilde{\deg}\varphi$ preserves the kernel of $\widetilde{\square}_\mu$.*

Proof. We need to show that if φ is a function on H_{ρ_0} satisfying $\widetilde{\square}_\mu\varphi = 0$, then $\widetilde{\square}_\mu((\mu^{-2} + N(X))^{1/2}\widetilde{\deg}\varphi) = 0$. For this purpose we compute the following commutators of operators:

$$\begin{aligned} [\square, (\mu^{-2} + N(X))^{1/2}] &= \square(\mu^{-2} + N(X))^{1/2} + 2 \sum_{i=1}^4 \left(\frac{\partial}{\partial v^i} (\mu^{-2} + N(X))^{1/2} \right) \cdot \frac{\partial}{\partial v^i} \\ &= \frac{4}{(\mu^{-2} + N(X))^{1/2}} - \frac{N(X)}{(\mu^{-2} + N(X))^{3/2}} + \frac{2}{(\mu^{-2} + N(X))^{1/2}} \deg, \end{aligned}$$

$$[\deg, (\mu^{-2} + N(X))^{1/2}] = \deg(\mu^{-2} + N(X))^{1/2} = N(X) \cdot (\mu^{-2} + N(X))^{-1/2},$$

$$\begin{aligned} [\deg^2, (\mu^{-2} + N(X))^{1/2}] &= \deg \circ [\deg, (\mu^{-2} + N(X))^{1/2}] + [\deg, (\mu^{-2} + N(X))^{1/2}] \circ \deg \\ &= \frac{2N(X)}{(\mu^{-2} + N(X))^{1/2}} - \frac{N(X)^2}{(\mu^{-2} + N(X))^{3/2}} + \frac{2N(X)}{(\mu^{-2} + N(X))^{1/2}} \deg, \end{aligned}$$

$$\begin{aligned} [\widetilde{\square}_\mu, (\mu^{-2} + N(X))^{1/2}] &= \\ (\mu^{-2} + N(X))^{-1/2} \left(4 - \frac{N(X)}{\mu^{-2} + N(X)} + 2\deg + \mu^2 \left(5N(X) - \frac{N(X)^2}{\mu^{-2} + N(X)} + 2N(X)\deg \right) \right) \\ &= 2\mu^2(\mu^{-2} + N(X))^{1/2}(\deg + 2). \end{aligned}$$

Finally, we get:

$$\begin{aligned} \widetilde{\square}_\mu((\mu^{-2} + N(X))^{1/2}\widetilde{\deg}\varphi) &= (\mu^{-2} + N(X))^{1/2}\widetilde{\square}_\mu\widetilde{\deg}\varphi + [\widetilde{\square}_\mu, (\mu^{-2} + N(X))^{1/2}]\widetilde{\deg}\varphi \\ &= (\mu^{-2} + N(X))^{1/2}(2\mu^2(\deg + 2)\widetilde{\deg}\varphi) = 2(\mu^{-2} + N(X))^{1/2}\widetilde{\square}_\mu\varphi = 0. \end{aligned}$$

\square

11 Extension of Harmonic Functions to $\mathbb{R}^{1,4}$

If we identify the group $SU(2)$ with the unit sphere in \mathbb{H} , then functions on $SU(2)$ can be extended to $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$ as harmonic functions. If we require such an extension to be regular either at the origin or at infinity, then it is unique. For example, let us consider the matrix coefficients of $SU(2)$ given by (14). The restrictions $t_{m\underline{n}}^l(X)|_{SU(2)}$ can be extended from $SU(2)$ to \mathbb{H}^\times as $t_{m\underline{n}}^l(X)$ which are homogeneous polynomials of degree $2l$ – hence regular at the origin – or as $N(X)^{-2l-1} \cdot t_{m\underline{n}}^l(X)$ which are homogeneous rational functions of degree $-2l-2$ – hence regular at infinity – and in both cases

$$\square t_{m\underline{n}}^l(X) = 0, \quad \square(t_{m\underline{n}}^l(X) \cdot N(X)^{-2l-1}) = 0.$$

In this section we start with a function on the unit sphere S^3 centered at the origin in \mathbb{R}^4 , realize \mathbb{R}^4 as a hyperplane $\{w^0 = \text{const}\}$ inside $\mathbb{R}^{1,4}$ and find the function's extensions to $\mathbb{R}_+^{1,4}$ which are annihilated both by the wave operator $\square_{1,4}$ and the conformal Laplacian $\tilde{\square}_\mu$. Let (r, \vec{n}) be the spherical coordinates of \mathbb{R}^4 spanned by w^1, w^2, w^3, w^4 , so that

$$r = \sqrt{(w^1)^2 + (w^2)^2 + (w^3)^2 + (w^4)^2} \quad \text{and} \quad \vec{n} \in S^3.$$

Then

$$\frac{\partial^2}{(\partial w^1)^2} + \frac{\partial^2}{(\partial w^2)^2} + \frac{\partial^2}{(\partial w^3)^2} + \frac{\partial^2}{(\partial w^4)^2} = \frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{\Delta_{S^3}}{r^2},$$

where Δ_{S^3} denotes the spherical Laplacian on the unit sphere in \mathbb{R}^4 . In particular, we obtain a set of coordinates on $\mathbb{R}_+^{1,4}$:

$$(w^0, w^1, w^2, w^3, w^4) \quad \longleftrightarrow \quad (w^0, r, \vec{n}).$$

We perform another change of coordinates

$$(w^0, r, \vec{n}) \quad \longleftrightarrow \quad (\rho, \theta, \vec{n})$$

with

$$w^0 = \rho \cosh \theta, \quad r = \rho \sinh \theta, \quad \rho = \sqrt{(w^0)^2 - r^2}, \quad \tanh \theta = r/w^0.$$

Then

$$\begin{aligned} \frac{\partial}{\partial w^0} &= \cosh \theta \frac{\partial}{\partial \rho} - \frac{\sinh \theta}{\rho} \frac{\partial}{\partial \theta}, & \frac{\partial}{\partial r} &= -\sinh \theta \frac{\partial}{\partial \rho} + \frac{\cosh \theta}{\rho} \frac{\partial}{\partial \theta}, \\ \frac{\partial^2}{(\partial w^0)^2} - \frac{\partial^2}{\partial r^2} &= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}, \end{aligned}$$

and it follows that

$$\square_{1,4} = \frac{\partial^2}{(\partial w^0)^2} - \frac{\partial^2}{\partial r^2} - \frac{3}{r} \frac{\partial}{\partial r} - \frac{\Delta_{S^3}}{r^2} = \frac{\partial^2}{\partial \rho^2} + \frac{4}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} - \frac{3 \cosh \theta}{\rho^2 \sinh \theta} \frac{\partial}{\partial \theta} - \frac{\Delta_{S^3}}{\rho^2 \sinh^2 \theta}.$$

Now we look for solutions of $\square_{1,4}\varphi(W) = 0$ in the separated form

$$\varphi(W) = \rho^\lambda \cdot s_l(\theta) \cdot t_{m\underline{n}}^l(\vec{n}),$$

where $t_{m\underline{n}}^l$'s are the matrix coefficients of $SU(2)$ defined by (14). Since $r^{2l} \cdot t_{m\underline{n}}^l(\vec{n})$ are harmonic homogeneous polynomials in w^1, \dots, w^4 of degree $2l$, it follows that the matrix coefficients $t_{m\underline{n}}^l$'s are eigenfunctions for Δ_{S^3} :

$$\Delta_{S^3} t_{m\underline{n}}^l(\vec{n}) = -4l(l+1) t_{m\underline{n}}^l(\vec{n}).$$

Moreover, any eigenfunction of Δ_{S^3} is a linear combination of $t_{m\underline{n}}^l$'s.

Since \square_μ does not depend on ρ , by (40),

$$\square_{1,4}(\rho^\lambda \cdot s_l(\theta) \cdot t_{m\underline{n}}^l(\vec{n})) = 0 \quad \Longleftrightarrow \quad (\square_\mu - \lambda(\lambda + 3)\mu^2)(s_l(\theta) \cdot t_{m\underline{n}}^l(\vec{n})) = 0.$$

Recall that we are looking for functions annihilated by $\tilde{\square}_\mu = \square_\mu + 2\mu^2$ as well. Hence

$$\lambda(\lambda + 3) + 2 = 0 \quad \text{and} \quad \lambda = -1 \quad \text{or} \quad \lambda = -2. \quad (42)$$

The equation $\square_{1,4}\varphi(W) = 0$ becomes an ordinary differential equation for $s_l(\theta)$:

$$\begin{aligned} 0 = \square_{1,4}\varphi(W) &= \left(\frac{\partial^2}{\partial \rho^2} + \frac{4}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} - \frac{3 \cosh \theta}{\rho^2 \sinh \theta} \frac{\partial}{\partial \theta} - \frac{\Delta_{S^3}}{\rho^2 \sinh^2 \theta} \right) \rho^\lambda \cdot s_l(\theta) \cdot t_{m\underline{n}}^l(\vec{n}) \\ &= \rho^{\lambda-2} \cdot t_{m\underline{n}}^l(\vec{n}) \cdot \left(\lambda(\lambda - 1) + 4\lambda - \frac{\partial^2}{\partial \theta^2} - 3 \frac{\cosh \theta}{\sinh \theta} \frac{\partial}{\partial \theta} + \frac{4l(l+1)}{\sinh^2 \theta} \right) s_l(\theta). \end{aligned}$$

Thus, the function $s_l(\theta)$ satisfies a differential equation

$$\left(\frac{d^2}{d\theta^2} + 3 \frac{\cosh \theta}{\sinh \theta} \frac{d}{d\theta} - \frac{4l(l+1)}{\sinh^2 \theta} - \lambda(\lambda + 3) \right) s_l(\theta) = 0.$$

Changing the variable θ to $t = \cosh \theta$ and using $\frac{d}{d\theta} = \sinh \theta \frac{d}{dt}$, we can rewrite this equation as

$$\left((t^2 - 1) \frac{d^2}{dt^2} + 4t \frac{d}{dt} - \frac{4l(l+1)}{t^2 - 1} - \lambda(\lambda + 3) \right) s_l(t) = 0.$$

But $\lambda(\lambda + 3) = -2$ by (42), so

$$\left((t^2 - 1) \frac{d^2}{dt^2} + 4t \frac{d}{dt} - \frac{4l(l+1)}{t^2 - 1} + 2 \right) s_l(t) = 0.$$

It is easy to verify directly that

$$\frac{(t-1)^l}{(t+1)^{l+1}} \quad \text{and} \quad \frac{(t+1)^l}{(t-1)^{l+1}}, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots,$$

are two linearly independent solutions of this equation. Thus we obtain four families of functions on $\mathbb{R}_+^{1,4}$ that simultaneously satisfy $\square_{1,4}\varphi = 0$ and $\tilde{\square}_\mu\varphi = 0$:

$$\rho^\lambda \cdot \frac{(\cosh \theta - 1)^l}{(\cosh \theta + 1)^{l+1}} \cdot t_{m\underline{n}}^l(\vec{n}) \quad \text{and} \quad \rho^\lambda \cdot \frac{(\cosh \theta + 1)^l}{(\cosh \theta - 1)^{l+1}} \cdot t_{m\underline{n}}^l(\vec{n}), \quad \begin{array}{l} \lambda = -1 \text{ or } \lambda = -2, \\ l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots \end{array}$$

Since $\mu^2 \rho^2 N(X) = r^2 = \rho^2 \sinh^2 \theta$, we have $\sinh^2 \theta = \mu^2 N(X)$ and $\cosh^2 \theta = 1 + \mu^2 N(X)$. Then we can rewrite our functions using

$$t_{m\underline{n}}^l(\vec{n}) = t_{m\underline{n}}^l(X) \cdot N(X)^{-l} = \mu^{2l} \cdot (\sinh \theta)^{-2l} \cdot t_{m\underline{n}}^l(X) = \mu^{2l} \cdot (\cosh \theta - 1)^{-l} \cdot (\cosh \theta + 1)^{-l} \cdot t_{m\underline{n}}^l(X).$$

We summarize the results of this section as a proposition.

Proposition 29. *We have four families of functions on $\mathbb{R}_+^{1,4}$ that simultaneously satisfy $\square_{1,4}\varphi = 0$ and $\widetilde{\square}_\mu\varphi = 0$*

$$\frac{\rho^\lambda \cdot t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+1}} \quad \text{and} \quad \frac{\rho^\lambda \cdot t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} - 1)^{2l+1}}, \quad \begin{array}{l} \lambda = -1 \text{ or} \\ \lambda = -2, \end{array} \quad (43)$$

where $\rho = \sqrt{(w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2 - (w^4)^2}$ and $X = (x^0, x^1, x^2, x^3) = (\frac{w^1}{\mu\rho}, \frac{w^2}{\mu\rho}, \frac{w^3}{\mu\rho}, \frac{w^4}{\mu\rho})$.

Up to proportionality coefficients, these functions extend the matrix coefficient functions $t_{m\underline{n}}^l(\vec{n})$ on

$$S^3 = \{W \in H_{\rho 0}; (w^1)^2 + (w^2)^2 + (w^3)^2 + (w^4)^2 = 1\} \subset \mathbb{R}_+^{1,4}.$$

Moreover, any other extension of $t_{m\underline{n}}^l(\vec{n})$ to $\mathbb{R}_+^{1,4}$ satisfying both $\square_{1,4}\varphi = 0$ and $\widetilde{\square}_\mu\varphi = 0$ is a linear combination of functions (43).

12 Spaces of Solutions of $\widetilde{\square}_\mu\varphi = 0$ and an Invariant Bilinear Pairing

We denote by $\overline{\mathcal{H}_\mu}$ the space of solutions of $\widetilde{\square}_\mu\varphi = 0$ on \mathbb{H}^\times ; these solutions are real analytic functions. We introduce algebraic subspaces

$$\begin{aligned} \mathcal{H}_\mu^+ &= \mathbb{C}\text{-span of } \frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+1}}, & l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, \\ & & m, n = -l, -l+1, \dots, l, \\ \mathcal{H}_\mu^- &= \mathbb{C}\text{-span of } \frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} - 1)^{2l+1}}, & l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, \\ & & m, n = -l, -l+1, \dots, l, \end{aligned}$$

and $\mathcal{H}_\mu = \mathcal{H}_\mu^+ \oplus \mathcal{H}_\mu^-$. Note that when $\mu \rightarrow 0$,

$$\frac{2^{2l+1} t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+1}} \rightarrow t_{m\underline{n}}^l(X)$$

and

$$\frac{2^{-2l-1} \mu^{4l+2} \cdot t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} - 1)^{2l+1}} \rightarrow t_{m\underline{n}}^l(X) \cdot N(X)^{-2l-1}.$$

Since the restrictions of $t_{m\underline{n}}^l(X) \cdot ((1 + \mu^2 N(X))^{1/2} \pm 1)^{-(2l+1)}$ to the unit ball in \mathbb{H} are dense in the space of all analytic functions on that ball, \mathcal{H}_μ is dense in $\overline{\mathcal{H}_\mu}$, justifying the notation. Taking closures we obtain a decomposition $\overline{\mathcal{H}_\mu} = \overline{\mathcal{H}_\mu^+} \oplus \overline{\mathcal{H}_\mu^-}$. The space \mathcal{H}_μ can be characterized as the space of all $SO(4)$ -finite solutions of $\widetilde{\square}_\mu\varphi = 0$ on \mathbb{H}^\times . Then \mathcal{H}_μ^+ can be characterized as the subspace of \mathcal{H}_μ consisting of functions that are regular at the origin. Finally, \mathcal{H}_μ^- can be characterized as the subspace of \mathcal{H}_μ consisting of functions that decay at infinity (or “regular at infinity”).

We introduce a bilinear pairing between $\overline{\mathcal{H}_\mu^+}$ and $\overline{\mathcal{H}_\mu^-}$:

$$\begin{aligned} (\varphi_1, \varphi_2)_\mu &= \frac{\sqrt{1 + \mu^2 R^2}}{2\pi^2} \int_{X \in S_R^3} (\widetilde{\deg \varphi_1})(X) \cdot \varphi_2(X) \frac{dS}{R} \\ &= -\frac{\sqrt{1 + \mu^2 R^2}}{2\pi^2} \int_{X \in S_R^3} \varphi_1(X) \cdot (\widetilde{\deg \varphi_2})(X) \frac{dS}{R}, \quad \varphi_1 \in \overline{\mathcal{H}_\mu^+}, \varphi_2 \in \overline{\mathcal{H}_\mu^-}, \end{aligned} \quad (44)$$

where S_R^3 denotes a sphere of radius $R > 0$ in \mathbb{H} centered at the origin and dS denotes the standard Euclidean measure on S_R^3 inherited from \mathbb{H} .

Proposition 30. *The two expressions in (44) agree; the resulting bilinear pairing is $SO^+(1, 4)$ -invariant, $\mathfrak{so}(6, \mathbb{C})$ -invariant, non-degenerate and independent of R . Moreover,*

$$\left(\frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+1}}, \frac{t_{n'm'}^{l'}(X^+)}{((1 + \mu^2 N(X))^{1/2} - 1)^{2l+1}} \right)_\mu = \frac{\delta_{ll'} \delta_{mm'} \delta_{nn'}}{\mu^{4l+2}}. \quad (45)$$

Proof. Using

$$\deg(1 + \mu^2 N(X))^{1/2} = \mu^2 N(X) \cdot (1 + \mu^2 N(X))^{-1/2}, \quad (46)$$

we obtain

$$\widetilde{\deg} \left(\frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} \pm 1)^{2l+1}} \right) = \frac{\pm(2l+1)t_{m\underline{n}}^l(X)}{(1 + \mu^2 N(X))^{1/2} \cdot ((1 + \mu^2 N(X))^{1/2} \pm 1)^{2l+1}}.$$

Then (45) follows from the orthogonality relations (17).

Since these basis functions are dense in $\overline{\mathcal{H}}_\mu^+$ and $\overline{\mathcal{H}}_\mu^-$ respectively, this computation proves that the two expressions in (44) agree, independent of $R > 0$ and the resulting bilinear pairing is non-degenerate. It remains to prove that it is $SO^+(1, 4)$ -invariant and $\mathfrak{so}(6, \mathbb{C})$ -invariant. The proof will be given in Corollary 33. \square

13 Poisson Formula

In this section we prove a Poisson-type formula for functions on \mathbb{H} annihilated by $\tilde{\square}_\mu$. As an intermediate step, we derive an expansion for $(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^{-1}$ similar to the matrix coefficient expansions for $N(X - Y)^{-1}$ given by Proposition 25 from [FL1] and restated here as equation (35). We recall the notations of Lemma 24: for $X, Y \in \mathbb{H}$, let

$$\hat{X} = (\sqrt{\mu^{-2} + N(X)}, x^0, x^1, x^2, x^3) \quad \text{and} \quad \hat{Y} = (\sqrt{\mu^{-2} + N(Y)}, y^0, y^1, y^2, y^3) \in \mathbb{R}_+^{1,4}.$$

Proposition 31. *We have the following expansion:*

$$-\frac{1}{\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4}} = \sum_{l,m,n} \frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+1}} \cdot \frac{\mu^{4l+2} \cdot t_{n\underline{m}}^l(Y^+)}{((1 + \mu^2 N(Y))^{1/2} - 1)^{2l+1}},$$

which converges pointwise absolutely in the region $\{(X, Y) \in \mathbb{H} \times \mathbb{H}; N(X) < N(Y)\}$. The sum is taken first over all $m, n = -l, -l+1, \dots, l$, then over $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$.

Proof. Let $X, Y \in \mathbb{H}$ and

$$\vec{u} = \frac{X}{\sqrt{N(X)}}, \quad \vec{v} = \frac{Y}{\sqrt{N(Y)}} \in SU(2) \subset \mathbb{H},$$

then $\vec{u} \vec{v}^{-1}$ is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Define θ_1, θ_2, t_1 and t_2 by

$$\sinh \theta_1 = \mu \sqrt{N(X)}, \quad \sinh \theta_2 = \mu \sqrt{N(Y)}, \quad t_1 = \cosh \theta_1, \quad t_2 = \cosh \theta_2. \quad (47)$$

Using the multiplicativity property of matrix coefficients (15), we compute:

$$\begin{aligned}
& \sum_{l,m,n} \frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+1}} \cdot \frac{\mu^{4l+2} \cdot t_{n\underline{m}}^l(Y^+)}{((1 + \mu^2 N(Y))^{1/2} - 1)^{2l+1}} \\
&= \mu^2 \sum_l \frac{(t_1 - 1)^l}{(t_1 + 1)^{l+1}} \cdot \frac{(t_2 + 1)^l}{(t_2 - 1)^{l+1}} \cdot \chi_l(\vec{u} \vec{v}^{-1}) \\
&= \frac{\mu^2}{\lambda - \lambda^{-1}} \sum_l \frac{(t_1 - 1)^l}{(t_1 + 1)^{l+1}} \cdot \frac{(t_2 + 1)^l}{(t_2 - 1)^{l+1}} \cdot (\lambda^{2l+1} - \lambda^{-2l-1}).
\end{aligned}$$

Let

$$a = (e^{\theta_1/2} + e^{-\theta_1/2})(e^{\theta_2/2} - e^{-\theta_2/2}), \quad b = (e^{\theta_1/2} - e^{-\theta_1/2})(e^{\theta_2/2} + e^{-\theta_2/2}), \quad (48)$$

then

$$\begin{aligned}
& \sum_{l,m,n} \frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+1}} \cdot \frac{\mu^{4l+2} \cdot t_{n\underline{m}}^l(Y^+)}{((1 + \mu^2 N(Y))^{1/2} - 1)^{2l+1}} \\
&= \frac{4\mu^2(\lambda - \lambda^{-1})^{-1}}{(e^{\theta_1} + e^{-\theta_1} + 2)(e^{\theta_2} + e^{-\theta_2} - 2)} \sum_l a^{-2l} \cdot b^{2l} \cdot (\lambda^{2l+1} - \lambda^{-2l-1}) \\
&= \frac{4\mu^2}{(\lambda - \lambda^{-1})a} \left(\frac{\lambda}{a - \lambda b} - \frac{\lambda^{-1}}{a - \lambda^{-1}b} \right) = \frac{4\mu^2}{(a - \lambda b)(a - \lambda^{-1}b)} = \frac{4\mu^2}{N(b\vec{u} - a\vec{v})}.
\end{aligned}$$

Since $X = \mu^{-1} \sinh \theta_1 \vec{u}$ and $Y = \mu^{-1} \sinh \theta_2 \vec{v}$,

$$\begin{aligned}
\frac{4\mu^2}{N(b\vec{u} - a\vec{v})} &= 4\mu^2 \left[N \left(\frac{\mu b X}{\sinh \theta_1} - \frac{\mu a Y}{\sinh \theta_2} \right) \right]^{-1} \\
&= \left[N \left(\frac{e^{\theta_2/2} + e^{-\theta_2/2}}{e^{\theta_1/2} + e^{-\theta_1/2}} X - \frac{e^{\theta_1/2} + e^{-\theta_1/2}}{e^{\theta_2/2} + e^{-\theta_2/2}} Y \right) \right]^{-1} = \frac{(\cosh \theta_1 + 1)(\cosh \theta_2 + 1)}{N((\cosh \theta_2 + 1)X - (\cosh \theta_1 + 1)Y)} \\
&= \frac{1/2}{\mu^{-2} \cosh \theta_1 \cosh \theta_2 - \mu^{-2} - (x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3)} \\
&= -\frac{1}{\langle \hat{X}, \hat{X} \rangle_{1,4} + \langle \hat{Y}, \hat{Y} \rangle_{1,4} - 2\langle \hat{X}, \hat{Y} \rangle_{1,4}} = -\frac{1}{\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4}}. \quad (49)
\end{aligned}$$

This expansion holds whenever $|b/a| < 1$. Since

$$\frac{b}{a} = \frac{\tanh(\theta_1/2)}{\tanh(\theta_2/2)} \geq 0$$

and $\tanh \theta$ is monotone increasing, the expansion holds whenever $\theta_1 < \theta_2$ or, equivalently, $N(X) < N(Y)$. \square

We introduce a notation

$$K_\mu(X, Y) = -\frac{1}{\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4}}.$$

Now we are ready to prove the Poisson-type formula. Let S_R^3 denote a sphere of radius R in \mathbb{H} centered at the origin.

Theorem 32. Let φ be a real analytic solution of $\widetilde{\square}_\mu \varphi = 0$ defined on a closed ball $\{X \in \mathbb{H}; N(X) \leq R^2\}$, for some $R > 0$. Then, for all $Y \in \mathbb{H}$ with $N(Y) < R^2$,

$$\begin{aligned}\varphi(Y) &= (\varphi(X), K_\mu(X, Y))_\mu \\ &= \frac{\sqrt{1 + \mu^2 R^2}}{2\pi^2} \int_{X \in S_R^3} (\widetilde{\deg \varphi})(X) \cdot K_\mu(X, Y) \frac{dS}{R} \\ &= -\frac{\sqrt{1 + \mu^2 R^2}}{2\pi^2} \int_{X \in S_R^3} (\widetilde{\deg_X K_\mu})(X, Y) \cdot \varphi(X) \frac{dS}{R}.\end{aligned}$$

Similarly, suppose φ is a real analytic solution of $\widetilde{\square}_\mu \varphi = 0$ defined on a closed set $\{X \in \mathbb{H}; N(X) \geq R^2\}$, for some $R > 0$, and regular at infinity. Then, for all $Y \in \mathbb{H}$ with $N(Y) > R^2$,

$$\begin{aligned}\varphi(Y) &= (K_\mu(X, Y), \varphi(X))_\mu \\ &= -\frac{\sqrt{1 + \mu^2 R^2}}{2\pi^2} \int_{X \in S_R^3} (\widetilde{\deg \varphi})(X) \cdot K_\mu(X, Y) \frac{dS}{R} \\ &= \frac{\sqrt{1 + \mu^2 R^2}}{2\pi^2} \int_{X \in S_R^3} (\widetilde{\deg_X K_\mu})(X, Y) \cdot \varphi(X) \frac{dS}{R}.\end{aligned}$$

Proof. It is sufficient to prove the formulas when

$$\varphi(X) = \frac{t_{m\underline{n}}^l(X)}{((1 + \mu^2 N(X))^{1/2} \pm 1)^{2l+1}}.$$

Then the result follows from the expansion of the kernel $K_\mu(X, Y)$ and orthogonality relations (45). \square

Corollary 33. The bilinear pairing (44) is $SO^+(1, 4)$ -invariant and $\mathfrak{so}(6, \mathbb{C})$ -invariant.

Proof. Since the group is connected, to prove $SO^+(1, 4)$ -invariance, it is sufficient to show invariance for $a \in SO^+(1, 4)$ sufficiently close to the identity only. Choose R_1, R_2 such that $0 < R_2 < R < R_1$ and, using the Poisson formula, write

$$\varphi_i(X) = (-1)^{i+1} \frac{\sqrt{1 + \mu^2 R_i^2}}{2\pi^2} \int_{Y_i \in S_{R_i}^3} (\widetilde{\deg \varphi_i})(Y_i) \cdot K_\mu(X, Y_i) \frac{dS}{R_i}, \quad i = 1, 2.$$

In short,

$$\varphi_1(X) = (\varphi_1(Y_1), K_\mu(X, Y_1))_\mu \quad \text{and} \quad \varphi_2(X) = (K_\mu(X, Y_2), \varphi_2(Y_2))_\mu.$$

Then

$$\begin{aligned}((\pi(a)\varphi_1)(X), (\pi(a)\varphi_2)(X))_\mu \\ &= \left((\varphi_1(Y_1), \pi(a)_X K_\mu(X, Y_1))_\mu, (\pi(a)_X K_\mu(X, Y_2), \varphi_2(Y_2))_\mu \right)_\mu \\ &= ((\varphi_1(Y_1), K_\mu(X, Y_1))_\mu, (K_\mu(X, Y_2), \varphi_2(Y_2))_\mu)_\mu = (\varphi_1(X), \varphi_2(X))_\mu\end{aligned}$$

because the Poisson formula is $SO^+(1, 4)$ -equivariant and

$$\begin{aligned}(\pi(a)_X K_\mu(X, Y_1), \pi(a)_X K_\mu(X, Y_2))_\mu \\ &= (\pi(a)_{Y_1} \circ \pi(a)_X K_\mu(X, Y_1), K_\mu(X, Y_2))_\mu = (K_\mu(X, Y_1), K_\mu(X, Y_2))_\mu.\end{aligned}$$

Since the action of $\mathfrak{so}(6, \mathbb{C})$ is generated by the actions of $\mathfrak{so}(1, 4)$ and the operator $\varphi \mapsto (1 + \mu^2 N(X))^{1/2} \widetilde{\deg} \varphi$, to prove $\mathfrak{so}(6, \mathbb{C})$ -invariance, it is sufficient to prove invariance under the operator $\varphi \mapsto (1 + \mu^2 N(X))^{1/2} \widetilde{\deg} \varphi$ only. By Lemma 24,

$$\begin{aligned} (1 + \mu^2 N(X))^{1/2} \widetilde{\deg}_X K_\mu(X, Y) &= \frac{2}{\mu^2} \frac{(1 + \mu^2 N(Y))^{1/2} - (1 + \mu^2 N(X))^{1/2}}{(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^2} \\ &= -(1 + \mu^2 N(Y))^{1/2} \widetilde{\deg}_Y K_\mu(X, Y), \end{aligned}$$

and then the proof continues in exactly the same way as for $SO^+(1, 4)$ -invariance. \square

14 Regular Functions

In order to define analogues of left and right regular functions, we need to factor $\widetilde{\square}_\mu$ as a product of two Dirac-like operators. The factorization

$$\square = \nabla \nabla^+ = \nabla^+ \nabla \quad \text{can be rewritten as} \quad \begin{pmatrix} 0 & \nabla^+ \\ \nabla & 0 \end{pmatrix}^2 = \begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix} = \square \cdot I_{2 \times 2},$$

and

$$\begin{pmatrix} 0 & \nabla^+ \\ \nabla & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \quad \Longleftrightarrow \quad \nabla f_1 = 0 \text{ and } \nabla^+ f_2 = 0,$$

i.e. f_2 is left-regular and f_1 is anti-left-regular.

Proposition 34. *Let*

$$\nabla_\mu = \begin{pmatrix} \mu(X \nabla - \widetilde{\deg}) & (1 + \mu^2 N(X))^{1/2} \nabla^+ \\ (1 + \mu^2 N(X))^{1/2} \nabla & \mu(X^+ \nabla^+ - \widetilde{\deg}) \end{pmatrix},$$

then we have a factorization

$$\widetilde{\square}_\mu = \nabla_\mu (\nabla_\mu - \mu).$$

Proof. We use the following identities:

$$\nabla (1 + \mu^2 N(X))^{1/2} = \frac{\mu^2 X^+}{(1 + \mu^2 N(X))^{1/2}}, \quad \nabla^+ (1 + \mu^2 N(X))^{1/2} = \frac{\mu^2 X}{(1 + \mu^2 N(X))^{1/2}}, \quad (50)$$

$$X^+ \nabla^+ + \nabla X = \nabla^+ X^+ + X \nabla = 2(2 + \deg), \quad (51)$$

the last identity in turn implies

$$(X \nabla - \widetilde{\deg})^2 = (X^+ \nabla^+ - \widetilde{\deg})^2 = \widetilde{\deg}^2 - N(X) \square. \quad (52)$$

First, we find that ∇_μ^2 is equal to

$$\begin{pmatrix} \mu^2 (X \nabla - \widetilde{\deg})^2 + & \mu (X \nabla - \widetilde{\deg}) (1 + \mu^2 N(X))^{1/2} \nabla^+ \\ (1 + \mu^2 N(X))^{1/2} \nabla^+ (1 + \mu^2 N(X))^{1/2} \nabla & + \mu (1 + \mu^2 N(X))^{1/2} \nabla^+ (X^+ \nabla^+ - \widetilde{\deg}) \\ \mu (X^+ \nabla^+ - \widetilde{\deg}) (1 + \mu^2 N(X))^{1/2} \nabla & \mu^2 (X^+ \nabla^+ - \widetilde{\deg})^2 + \\ + \mu (1 + \mu^2 N(X))^{1/2} \nabla (X \nabla - \widetilde{\deg}) & (1 + \mu^2 N(X))^{1/2} \nabla (1 + \mu^2 N(X))^{1/2} \nabla^+ \end{pmatrix}$$

and then work out each entry separately. By (50) and (52), the diagonal terms are

$$\begin{aligned}\mu^2(\widetilde{\deg}^2 - N(X)\square) + (1 + \mu^2 N(X))\square + \mu^2 X\nabla &= \square + \mu^2(\widetilde{\deg}^2 + X\nabla), \\ \mu^2(\widetilde{\deg}^2 - N(X)\square) + (1 + \mu^2 N(X))\square + \mu^2 X^+ \nabla^+ &= \square + \mu^2(\widetilde{\deg}^2 + X^+ \nabla^+).\end{aligned}$$

By (46), (50) and (51), the off-diagonal terms are

$$\begin{aligned}\mu(X\nabla - \widetilde{\deg})(1 + \mu^2 N(X))^{1/2} \nabla^+ + \mu(1 + \mu^2 N(X))^{1/2} \nabla^+(X^+ \nabla^+ - \widetilde{\deg}) \\ = \mu(1 + \mu^2 N(X))^{1/2} (X\square - \widetilde{\deg} \nabla^+ + \nabla^+ X^+ \nabla^+ - \nabla^+ \widetilde{\deg}) = \mu(1 + \mu^2 N(X))^{1/2} \nabla^+\end{aligned}$$

and similarly

$$\begin{aligned}\mu(X^+ \nabla^+ - \widetilde{\deg})(1 + \mu^2 N(X))^{1/2} \nabla + \mu(1 + \mu^2 N(X))^{1/2} \nabla(X\nabla - \widetilde{\deg}) \\ = \mu(1 + \mu^2 N(X))^{1/2} (X^+ \square - \widetilde{\deg} \nabla + \nabla X\nabla - \nabla \widetilde{\deg}) = \mu(1 + \mu^2 N(X))^{1/2} \nabla.\end{aligned}$$

This proves $\nabla_\mu^2 = \widetilde{\square}_\mu + \mu \nabla_\mu$. □

The two equations (51) can be combined into a single equation as

$$\begin{pmatrix} 0 & \nabla^+ \\ \nabla & 0 \end{pmatrix} \begin{pmatrix} 0 & X \\ X^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & X \\ X^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & \nabla^+ \\ \nabla & 0 \end{pmatrix} = 2 \begin{pmatrix} \deg+2 & 0 \\ 0 & \deg+2 \end{pmatrix} = 2(\deg+2).$$

In our context this formula becomes

$$(\nabla_\mu - \mu) \begin{pmatrix} 0 & X \\ X^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & X \\ X^+ & 0 \end{pmatrix} \nabla_\mu = 2(1 + \mu^2 N(X))^{1/2} (\deg+2).$$

We introduce another operator, $\overleftarrow{\nabla}_\mu$, which we apply to functions on the right:

$$\begin{aligned}\overleftarrow{\nabla}_\mu : (g_1, g_2) &\mapsto (g_1, g_2) \begin{pmatrix} \mu(\widetilde{\deg} - \nabla^+ X^+) & \nabla^+(1 + \mu^2 N(X))^{1/2} \\ \nabla(1 + \mu^2 N(X))^{1/2} & \mu(\widetilde{\deg} - \nabla X) \end{pmatrix} \\ &= \begin{pmatrix} \mu(\widetilde{\deg} g_1 - (g_1 \nabla^+) X^+) + (1 + \mu^2 N(X))^{1/2} (g_2 \nabla) \\ (1 + \mu^2 N(X))^{1/2} (g_1 \nabla^+) + \mu(\widetilde{\deg} g_2 - (g_2 \nabla) X) \end{pmatrix}.\end{aligned}$$

Proposition 35. *We have a factorization*

$$\widetilde{\square}_\mu = \overleftarrow{\nabla}_\mu (\overleftarrow{\nabla}_\mu + \mu).$$

Remark 36. *One can produce functions satisfying $\nabla_\mu f = 0$ and $g \overleftarrow{\nabla}_\mu = 0$ as follows. Start with a \mathbb{C} -valued function φ annihilated by $\widetilde{\square}_\mu$ (for example, take $\varphi \in \mathcal{H}_\mu$). Then the two columns of $(\nabla_\mu - \mu)\varphi$ satisfy $\nabla_\mu f = 0$ and the two rows of $\varphi(\overleftarrow{\nabla}_\mu + \mu)$ satisfy $g \overleftarrow{\nabla}_\mu = 0$.*

15 Basic Properties of Regular Functions

Recall from [FL1] that

$$Dx = dx^1 \wedge dx^2 \wedge dx^3 - idx^0 \wedge dx^2 \wedge dx^3 + jdx^0 \wedge dx^1 \wedge dx^3 - kdx^0 \wedge dx^1 \wedge dx^2$$

is an \mathbb{H} -valued 3-form on \mathbb{H} that is Hodge dual to

$$dX = dx^0 + idx^1 + jdx^2 + kdx^3.$$

We also introduce

$$Dx^+ = dx^1 \wedge dx^2 \wedge dx^3 + idx^0 \wedge dx^2 \wedge dx^3 - jdx^0 \wedge dx^1 \wedge dx^3 + kdx^0 \wedge dx^1 \wedge dx^2,$$

which is Hodge dual to

$$dX^+ = dx^0 - idx^1 - jdx^2 - kdx^3$$

and

$$Dr = x^0 dx^1 \wedge dx^2 \wedge dx^3 - x^1 dx^0 \wedge dx^2 \wedge dx^3 + x^2 dx^0 \wedge dx^1 \wedge dx^3 - x^3 dx^0 \wedge dx^1 \wedge dx^2,$$

which is Hodge dual to

$$dr = x^0 dx^0 + x^1 dx^1 + x^2 dx^2 + x^3 dx^3.$$

Recall that $dV = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is the volume form on \mathbb{H} . If $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = (g_1, g_2)$ are two functions defined on an open set in \mathbb{H} , then

$$\begin{aligned} d(Dx \cdot f) &= (\nabla^+ f) dV, & d(Dx^+ \cdot f) &= (\nabla f) dV, & d(f \cdot Dr) &= (\deg f + 4f) dV, \\ d(g \cdot Dx) &= (g \nabla^+) dV, & d(g \cdot Dx^+) &= (g \nabla) dV. \end{aligned}$$

Note that

$$Dr = \frac{1}{2}(X \cdot Dx^+ + Dx \cdot X^+) = \frac{1}{2}(X^+ \cdot Dx + Dx^+ \cdot X).$$

Consider a matrix-valued 3-form on \mathbb{H}

$$Dx_\mu = \begin{pmatrix} \mu \frac{X \cdot Dx^+ - Dr}{(1 + \mu^2 N(X))^{1/2}} & Dx \\ Dx^+ & \mu \frac{X^+ \cdot Dx - Dr}{(1 + \mu^2 N(X))^{1/2}} \end{pmatrix} = \begin{pmatrix} \frac{\mu}{2} \frac{X \cdot Dx^+ - Dx \cdot X^+}{(1 + \mu^2 N(X))^{1/2}} & Dx \\ Dx^+ & \frac{\mu}{2} \frac{X^+ \cdot Dx - Dx^+ \cdot X}{(1 + \mu^2 N(X))^{1/2}} \end{pmatrix}.$$

Lemma 37.

$$d(g \cdot Dx_\mu \cdot f) = (1 + \mu^2 N(X))^{-1/2} ((g \overleftarrow{\nabla}_\mu) f + g(\nabla_\mu f)) dV.$$

Proof. Using

$$[X \nabla - \deg, (1 + \mu^2 N(X))^\alpha] = [X^+ \nabla^+ - \deg, (1 + \mu^2 N(X))^\alpha] = 0,$$

we compute the components of $d(g \cdot Dx_\mu \cdot f)$ coming from the diagonal entries of Dx_μ :

$$\begin{aligned} d(g_1(X \cdot Dx^+ - Dx \cdot X^+) f_1) &= \left(((g_1 X) \nabla - (g_1 \nabla^+) X^+) \cdot f_1 + g_1 \cdot (X(\nabla f_1) - \nabla^+(X^+ f_1)) \right) dV \\ &= 2 \left((\deg g_1 - (g_1 \nabla^+) X^+) \cdot f_1 + g_1 \cdot (X(\nabla f_1) - \deg f_1) \right) dV, \end{aligned}$$

$$\begin{aligned} d(g_2(X^+ \cdot Dx - Dx^+ \cdot X) f_2) &= \left(((g_2 X^+) \nabla^+ - (g_2 \nabla) X) \cdot f_2 + g_2 \cdot (X^+(\nabla^+ f_2) - \nabla(X f_2)) \right) dV \\ &= 2 \left((\deg g_2 - (g_2 \nabla) X) \cdot f_2 + g_2 \cdot (X^+(\nabla^+ f_2) - \deg f_2) \right) dV. \end{aligned}$$

Then the result follows. \square

As an immediate consequence we obtain Cauchy's integral theorem:

Corollary 38. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = (g_1, g_2)$ be two functions defined on an open set $U \subset \mathbb{H}$ such that $\nabla_\mu f = 0$ and $g \overleftarrow{\nabla}_\mu = 0$. Then $g \cdot Dx_\mu \cdot f$ is a closed 3-form. In particular, if C is a 3-cycle in U (with compact support), then the integral $\int_C g \cdot Dx_\mu \cdot f$ depends only on the homology class of C .

Lemma 39. Let $a, d \in \mathbb{H}$ with $N(a) = N(d) = 1$, then the pull-back of Dx_μ under the map $X \mapsto aXd^{-1}$ is

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} Dx_\mu \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{-1}.$$

Lemma 40. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be a left-regular function (i.e. satisfying $\nabla_\mu f = 0$). Then so is

$$\begin{pmatrix} a^{-1}f_1(aXd^{-1}) \\ d^{-1}f_2(aXd^{-1}) \end{pmatrix}, \quad \text{for any } a, d \in \mathbb{H} \text{ with } N(a) = N(d) = 1.$$

Proof. Using

$$\nabla(a^{-1}f_1(aXd^{-1})) = d^{-1}(\nabla f_1)|_{aXd^{-1}} \quad \text{and} \quad \nabla^+(d^{-1}f_2(aXd^{-1})) = a^{-1}(\nabla^+ f_2)|_{aXd^{-1}},$$

we compute

$$\nabla_\mu \begin{pmatrix} a^{-1}f_1(aXd^{-1}) \\ d^{-1}f_2(aXd^{-1}) \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{-1} \nabla_\mu \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \Big|_{aXd^{-1}} = 0.$$

□

16 Analogue of the Cauchy-Fueter Formula

The Cauchy-Fueter kernel in our setting is

$$k_\mu(X, Y) = -\frac{1}{2}K_\mu(X, Y)(\overleftarrow{\nabla}_\mu + \mu).$$

If $U \subset \mathbb{H}$ is an open region with piecewise \mathcal{C}^1 boundary ∂U , we define a preferred orientation on ∂U as follows. The positive orientation of U is determined by the vectors $\{1, i, j, k\}$ (or the volume form dV). Pick a non-singular point $p \in \partial U$ and let \vec{n}_p be a non-zero vector in $T_p\mathbb{H}$ perpendicular to $T_p\partial U$ and pointing outside of U . Then $\{\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3\} \subset T_p\partial U$ is positively oriented in ∂U if and only if $\{\vec{n}_p, \vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3\}$ is positively oriented in \mathbb{H} . Now we can prove our analogue of the Cauchy-Fueter formula:

Theorem 41. Let $U \subset \mathbb{H}$ be an open bounded subset with piecewise \mathcal{C}^1 boundary ∂U . Suppose that $f(X)$ is left-regular on a neighborhood of the closure \overline{U} (i.e. satisfying $\nabla_\mu f = 0$), then

$$\frac{1}{2\pi^2} \int_{\partial U} k_\mu(X, Y) \cdot Dx_\mu \cdot f(X) = \begin{cases} f(Y) & \text{if } Y \in U; \\ 0 & \text{if } Y \notin \overline{U}. \end{cases}$$

Remark 42. There is a similar formula for right-regular functions (i.e. functions satisfying $g \overleftarrow{\nabla}_\mu = 0$). The Cauchy-Fueter kernel in that case is $k'_\mu(X, Y) = -\frac{1}{2}(\nabla_\mu - \mu)K_\mu(X, Y)$.

Proof. By Remark 36 and Corollary 38 the integrand is a closed 3-form. If $Y \notin \overline{U}$, by Corollary 38 the integral is zero. So let us assume $Y \in U$. Consider a sphere $S_\varepsilon^3(Y)$ of radius ε centered at Y , and let ε be small enough so that the closed ball of radius ε centered at Y lies inside U . Then

$$\int_{\partial U} k_\mu(X, Y) \cdot Dx_\mu \cdot f(X) = \int_{S_\varepsilon^3(Y)} k_\mu(X, Y) \cdot Dx_\mu \cdot f(X).$$

The right hand side is independent from ε and it is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{S_\varepsilon^3(Y)} k_\mu(X, Y) \cdot Dx_\mu \cdot f(X) = 2\pi^2 \cdot f(Y).$$

We compute

$$\begin{aligned} & k_\mu(X, Y) \cdot (\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^2 \\ &= \begin{pmatrix} \mu Y X^+ & X \sqrt{1 + \mu^2 N(Y)} - Y \sqrt{1 + \mu^2 N(X)} \\ X^+ \sqrt{1 + \mu^2 N(Y)} - Y^+ \sqrt{1 + \mu^2 N(X)} & \mu Y^+ X \end{pmatrix} \\ &\quad + 2\mu^{-1} + \frac{\mu}{2} \text{Tr}(XY^+) - 2\mu^{-1} \sqrt{1 + \mu^2 N(X)} \cdot \sqrt{1 + \mu^2 N(Y)}. \end{aligned}$$

Let $X' = X - Y$, then $N(X') = \varepsilon^2$ and by Lemma 6 of [FL1]

$$Dx_\mu|_{S_\varepsilon^3(Y)} = \begin{pmatrix} \frac{\mu}{2} \frac{XX'^+ - X'X^+}{\sqrt{1 + \mu^2 N(X)}} & X' \\ X'^+ & \frac{\mu}{2} \frac{X^+X' - X'^+X}{\sqrt{1 + \mu^2 N(X)}} \end{pmatrix} \frac{dS}{\varepsilon}.$$

To simplify upcoming expressions, we introduce a notation $b = \sqrt{1 + \mu^2 N(Y)}$. Working with the lowest order terms with respect to X' and ignoring the higher order terms we get:

$$f(X) \sim f(Y), \quad (\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^2 \sim \varepsilon^4 \cdot \left(1 - \frac{\mu^2}{4\varepsilon^2 b^2} (\text{Tr}(X'Y^+))^2\right)^2,$$

$$Dx_\mu|_{S_\varepsilon^3(Y)} \sim \varepsilon^{-1} \cdot \begin{pmatrix} \frac{\mu}{2b} (YX'^+ - X'Y^+) & X' \\ X'^+ & \frac{\mu}{2b} (Y^+X' - X'^+Y) \end{pmatrix} dS,$$

$$k_\mu(X, Y) \sim$$

$$\varepsilon^{-4} \cdot \left(1 - \frac{\mu^2}{4\varepsilon^2 b^2} (\text{Tr}(X'Y^+))^2\right)^{-2} \cdot \begin{pmatrix} \mu Y X'^+ - \frac{\mu}{2} \text{Tr}(X'Y^+) & bX' - \frac{\mu^2}{2b} Y \cdot \text{Tr}(X'Y^+) \\ bX'^+ - \frac{\mu^2}{2b} Y^+ \cdot \text{Tr}(X'Y^+) & \mu Y^+ X' - \frac{\mu}{2} \text{Tr}(X'Y^+) \end{pmatrix}.$$

Using Lemma 40 we can assume that Y is real. Then

$$Dx_\mu|_{S_\varepsilon^3(Y)} \sim \varepsilon^{-1} \cdot \begin{pmatrix} -\frac{\mu}{b} Y \cdot \text{Im}(X') & X' \\ X'^+ & \frac{\mu}{b} Y \cdot \text{Im}(X') \end{pmatrix} dS,$$

$$k_\mu(X, Y) \sim$$

$$\varepsilon^{-4} \cdot \left(1 - \frac{\mu^2 Y^2}{\varepsilon^2 b^2} (\text{Re}(X'))^2\right)^{-2} \cdot \begin{pmatrix} -\mu Y \cdot \text{Im}(X') & bX' - \frac{\mu^2}{b} Y^2 \cdot \text{Re}(X') \\ bX'^+ - \frac{\mu^2}{b} Y^2 \cdot \text{Re}(X') & \mu Y \cdot \text{Im}(X') \end{pmatrix},$$

$$k_\mu(X, Y) \cdot Dx_\mu|_{S_\varepsilon^3(Y)} \sim$$

$$\begin{aligned} & \varepsilon^{-5} \cdot \left(1 - \frac{\mu^2 Y^2}{\varepsilon^2 b^2} (\text{Re}(X'))^2\right)^{-2} \cdot \left(\varepsilon^2 b + \frac{\mu^2 Y^2}{b} ((\text{Im}(X'))^2 - (\text{Re}(X'))^2)\right) dS \\ & + \varepsilon^{-5} \cdot \left(1 - \frac{\mu^2 Y^2}{\varepsilon^2 b^2} (\text{Re}(X'))^2\right)^{-2} \cdot \frac{\mu^2 Y^2}{b^2} \text{Re}(X') \cdot \text{Im}(X') \begin{pmatrix} b & -\mu Y \\ \mu Y & -b \end{pmatrix} dS. \end{aligned}$$

The integral over $S_\varepsilon^3(Y)$ of the last term is zero, and the first term simplifies to

$$\varepsilon^{-3} \cdot b^{-1} \left(1 - \frac{\mu^2 Y^2}{\varepsilon^2 b^2} (\operatorname{Re}(X'))^2 \right)^{-2} dS.$$

We finish the proof by integrating in spherical coordinates and using an integral

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{\sin^2 \theta d\theta}{(1 - a \cos^2 \theta)^2} = \frac{\pi}{2\sqrt{1-a}}, \quad |a| < 1,$$

with $a = \mu^2 Y^2 b^{-2}$. □

17 Deformation of \mathcal{H} and the Second Order Pole

Similarly to how we did in Section 12, we introduce a space of functions \mathcal{H}_μ , which is a deformation of \mathcal{H} . Then we discuss the analogues of the second order pole formulas given in Corollary 14 and Theorem 15. Thus we introduce a vector space

$$\mathcal{H}_\mu = \mathbb{C}\text{-span of } \frac{t_{m\underline{n}}^l(X) \cdot ((1 + \mu^2 N(X))^{1/2} - 1)^k}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+k+2}}, \quad \begin{array}{l} k \in \mathbb{Z}, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \\ m, n = -l, -l+1, \dots, l. \end{array}$$

Note that when $\mu \rightarrow 0$,

$$2^{2l+2k+2} \mu^{-2k} \frac{t_{m\underline{n}}^l(X) \cdot ((1 + \mu^2 N(X))^{1/2} - 1)^k}{((1 + \mu^2 N(X))^{1/2} + 1)^{2l+k+2}} \rightarrow t_{m\underline{n}}^l(X) \cdot N(X)^k.$$

We can extend these functions to an open neighborhood of \mathbb{H}^\times in $\mathbb{H}_\mathbb{C}^\times$ as follows. (We exclude $Z \in \mathbb{H}_\mathbb{C}$ such that $N(Z) = 0$ because $(1 + \mu^2 N(X))^{1/2} - 1$ vanishes there.) The matrix coefficient functions $t_{m\underline{n}}^l(X)$'s are polynomials in X , hence extend to $\mathbb{H}_\mathbb{C}$ without any problem. The only obstacle to extending the functions spanning \mathcal{H}_μ is the square root in $(1 + \mu^2 N(X))^{1/2} \pm 1$. Thus we choose the branch of $z^{1/2}$ defined on the complex plane without the negative real axis and observe that the functions

$$f_{k,l,m,n}(Z) = \frac{t_{m\underline{n}}^l(Z) \cdot ((1 + \mu^2 N(Z))^{1/2} - 1)^k}{((1 + \mu^2 N(Z))^{1/2} + 1)^{2l+k+2}}, \quad Z \in \mathbb{H}_\mathbb{C},$$

are well defined as long as $N(Z) \notin (-\infty, -\mu^{-2}]$ and $N(Z) \neq 0$. For this reason we introduce an open region in $\mathbb{H}_\mathbb{C}$

$$\mathbb{U}_\mu = \{Z \in \mathbb{H}_\mathbb{C}^\times; N(Z) \notin (-\infty, -\mu^{-2}]\}.$$

Our next task is to define a natural bilinear pairing on \mathcal{H}_μ . Fix an $R > 0$ and parameterize $U(2)_R$ as in Chapter III, §1, of [V]:

$$\begin{aligned} Z(\alpha, \varphi, \theta, \psi) &= R e^{i\alpha} \begin{pmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\ i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \cos \frac{\theta}{2} \cdot e^{i\frac{\varphi+\psi}{2}} & i \sin \frac{\theta}{2} \cdot e^{i\frac{\varphi-\psi}{2}} \\ i \sin \frac{\theta}{2} \cdot e^{i\frac{\psi-\varphi}{2}} & \cos \frac{\theta}{2} \cdot e^{-i\frac{\varphi+\psi}{2}} \end{pmatrix}, \end{aligned} \quad \begin{array}{l} 0 \leq \alpha < \pi, \\ 0 \leq \varphi < 2\pi, \\ 0 < \theta < \pi, \\ -2\pi \leq \psi < 2\pi. \end{array}$$

By direct computation we find

$$\begin{aligned} dV \Big|_{U(2)_R} &= dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3 \Big|_{U(2)_R} \\ &= \frac{1}{4} dz_{11} \wedge dz_{12} \wedge dz_{21} \wedge dz_{22} \Big|_{U(2)_R} = \frac{R^4}{8i} e^{4i\alpha} \sin \theta d\alpha \wedge d\varphi \wedge d\theta \wedge d\psi. \end{aligned}$$

For $0 < R < \mu^{-1}$, define a measure on $U(2)_R$ by

$$dV_{R,\mu} = \frac{R^4}{16} e^{4i\alpha} \sin \theta d\varphi \wedge d\theta \wedge d\psi \wedge d \log \left(\frac{(1 + \mu^2 R^2 e^{2i\alpha})^{1/2} - 1}{(1 + \mu^2 R^2 e^{2i\alpha})^{1/2} + 1} \right)$$

and define a bilinear pairing on \mathcal{H}_μ as

$$\langle f_1, f_2 \rangle_\mu = \frac{i}{2\pi^3} \int_{Z \in U(2)_R} f_1(Z) \cdot f_2(Z) dV_{R,\mu}, \quad f_1, f_2 \in \mathcal{H}_\mu. \quad (53)$$

(The parameter R is restricted to $0 < R < \mu^{-1}$ so that $U(2)_R \subset \mathbb{U}_\mu$.) We have the following analogue of the orthogonality relations (19):

Proposition 43. *The symmetric pairing (53) is independent of the choice of R (as long as $0 < R < \mu^{-1}$) and non-degenerate. Let*

$$f'_{k,l,m,n}(Z) = \frac{t_{n\underline{m}}^l(Z^+) \cdot ((1 + \mu^2 N(Z))^{1/2} + 1)^k}{((1 + \mu^2 N(Z))^{1/2} - 1)^{2l+k+2}} \in \mathcal{H}_\mu,$$

then we have orthogonality relations

$$\langle f_{k,l,m,n}(Z), f'_{k',l',m',n'}(Z) \rangle_\mu = \frac{\mu^{-4l-4}}{2l+1} \delta_{kk'} \delta_{ll'} \delta_{mm'} \delta_{nn'}, \quad (54)$$

where the indices k, l, m, n are $k \in \mathbb{Z}$, $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, $m, n \in \mathbb{Z} + l$, $-l \leq m, n \leq l$ and similarly for k', l', m', n' .

Proof. Since each family of functions $f_{k,l,m,n}(Z)$'s and $f'_{k,l,m,n}(Z)$'s generates \mathcal{H}_μ , the independence of R and non-degeneracy of the bilinear pairing follow from the orthogonality relations (54). Using the orthogonality relations (17), we obtain

$$\begin{aligned} & -2\pi^3 i \cdot \langle f_{k,l,m,n}(Z), f'_{k',l',m',n'}(Z) \rangle_\mu \\ &= \int_{Z \in U(2)_R} \frac{t_{m\underline{n}}^l(Z) \cdot ((1 + \mu^2 N(Z))^{1/2} - 1)^k}{((1 + \mu^2 N(Z))^{1/2} + 1)^{2l+k+2}} \cdot \frac{t_{n'\underline{m}'}^{l'}(Z^+) \cdot ((1 + \mu^2 N(Z))^{1/2} + 1)^{k'}}{((1 + \mu^2 N(Z))^{1/2} - 1)^{2l'+k'+2}} dV_{R,\mu} \\ &= \int \frac{\mu^{-4} \cdot t_{m\underline{n}}^l(Z) \cdot t_{n'\underline{m}'}^{l'}(Z^+) \cdot N(Z)^{-2}}{((1 + \mu^2 N(Z))^{1/2} + 1)^{2l} \cdot ((1 + \mu^2 N(Z))^{1/2} - 1)^{2l'}} \left(\frac{(1 + \mu^2 N(Z))^{1/2} - 1}{(1 + \mu^2 N(Z))^{1/2} + 1} \right)^{k-k'} dV_{R,\mu} \\ &= -\pi^2 \frac{\mu^{-4l-4}}{2l+1} \delta_{ll'} \delta_{mm'} \delta_{nn'} \int_{\alpha=0}^{\alpha=\pi} \left(\frac{(1 + \mu^2 R^2 e^{2i\alpha})^{1/2} - 1}{(1 + \mu^2 R^2 e^{2i\alpha})^{1/2} + 1} \right)^{k-k'} d \log \left(\frac{(1 + \mu^2 R^2 e^{2i\alpha})^{1/2} - 1}{(1 + \mu^2 R^2 e^{2i\alpha})^{1/2} + 1} \right) \\ &= -\pi^2 \frac{\mu^{-4l-4}}{2l+1} \delta_{ll'} \delta_{mm'} \delta_{nn'} \oint z^{k-k'-1} dz = -2\pi^3 i \frac{\mu^{-4l-4}}{2l+1} \delta_{kk'} \delta_{ll'} \delta_{mm'} \delta_{nn'}. \end{aligned}$$

□

Let us recall Proposition 27 from [FL1] restated here as Proposition 10. We want to obtain a similar expansion for $(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^{-2}$. We proceed as in the proof of Proposition 31. Let $X, Y \in \mathbb{U}_\mu$, let

$$t_1 = \sqrt{1 + \mu^2 N(X)}, \quad t_2 = \sqrt{1 + \mu^2 N(Y)} \quad \in \mathbb{C}$$

and choose $\theta_1, \theta_2 \in \mathbb{C}$ so that

$$\cosh \theta_1 = t_1 = \sqrt{1 + \mu^2 N(X)} \quad \text{and} \quad \cosh \theta_2 = t_2 = \sqrt{1 + \mu^2 N(Y)}.$$

(The square roots $\sqrt{1 + \mu^2 N(X)}$ and $\sqrt{1 + \mu^2 N(Y)}$ are uniquely defined, but θ_1 and θ_2 are not.) Then $\sinh^2 \theta_1 = \mu^2 N(X)$ and $\sinh^2 \theta_2 = \mu^2 N(Y)$. Define

$$\vec{u} = \frac{\mu X}{\sinh \theta_1}, \quad \vec{v} = \frac{\mu Y}{\sinh \theta_2} \quad \in \mathbb{H}_\mathbb{C},$$

and suppose that $\vec{u} \vec{v}^{-1}$ is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{C}$. Using the multiplicativity property of matrix coefficients (15) and our previous notations (48), we compute a sum over all $m, n = -l, -l+1, \dots, l$, then over $k = 0, 1, 2, 3, \dots$ and $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$:

$$\begin{aligned} & \sum_{k,l,m,n} (2l+1) \mu^{4l+4} f_{k,l,m,n}(X) \cdot f'_{k,l,m,n}(Y) \\ &= \mu^4 \sum_{k,l} (2l+1) \frac{(t_1-1)^{l+k}}{(t_1+1)^{l+k+2}} \cdot \frac{(t_2+1)^{l+k}}{(t_2-1)^{l+k+2}} \cdot \chi_l(\vec{u} \vec{v}^{-1}) \\ &= \frac{16\mu^4 (\lambda - \lambda^{-1})^{-1}}{(e^{\theta_1} + e^{-\theta_1} + 2)^2 (e^{\theta_2} + e^{-\theta_2} - 2)^2} \sum_{k,l} (2l+1) \frac{b^{2l+2k}}{a^{2l+2k}} \cdot (\lambda^{2l+1} - \lambda^{-2l-1}) \\ &= \frac{16\mu^4 (\lambda - \lambda^{-1})^{-1}}{(e^{\theta_1/2} + e^{-\theta_1/2})^4 (e^{\theta_2/2} - e^{-\theta_2/2})^4 (1 - b^2/a^2)} \sum_l (2l+1) \frac{b^{2l}}{a^{2l}} \cdot (\lambda^{2l+1} - \lambda^{-2l-1}) \\ &= \frac{16\mu^4}{(a - \lambda b)^2 (a - \lambda^{-1} b)^2} = \frac{16\mu^4}{N^2(b\vec{u} - a\vec{v})} = \frac{1}{(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^2}, \quad (55) \end{aligned}$$

where in the last step we used (49) and

$$\hat{X} = (\sqrt{\mu^{-2} + N(X)}, x^0, x^1, x^2, x^3) \quad \text{and} \quad \hat{Y} = (\sqrt{\mu^{-2} + N(Y)}, y^0, y^1, y^2, y^3) \quad \in \mathbb{R}_+^{1,4}.$$

Like the expansion of $(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^{-1}$, this expansion holds whenever

$$|\lambda b/a| < 1 \quad \text{and} \quad |\lambda^{-1} b/a| < 1$$

and, in particular, for those X, Y the denominator does not turn to zero.

We avoid finding the region where these inequalities are satisfied and impose instead an assumption that $|\lambda| = 1$. We have:

$$\frac{b^2}{a^2} = \frac{\tanh^2(\theta_1/2)}{\tanh^2(\theta_2/2)} = \frac{t_1 - 1}{t_1 + 1} \frac{t_2 + 1}{t_2 - 1} = \frac{\sqrt{1 + \mu^2 N(X)} - 1}{\sqrt{1 + \mu^2 N(X)} + 1} \frac{\sqrt{1 + \mu^2 N(Y)} + 1}{\sqrt{1 + \mu^2 N(Y)} - 1}.$$

Thus the expansion (55) certainly holds for $X, Y \in \mathbb{U}_\mu$ such that XY^{-1} is diagonalizable with both eigenvalues having the same length and

$$\left| \frac{\sqrt{1 + \mu^2 N(X)} - 1}{\sqrt{1 + \mu^2 N(X)} + 1} \right| < \left| \frac{\sqrt{1 + \mu^2 N(Y)} - 1}{\sqrt{1 + \mu^2 N(Y)} + 1} \right|.$$

The condition that XY^{-1} is diagonalizable with both eigenvalues having the same length is automatically satisfied if $X \in U(2)_{R_1}$ and $Y \in U(2)_{R_2}$.

Using single variable calculus, we can prove:

Lemma 44. *Let $0 < R < \mu^{-1}$. Then, as X ranges over $U(2)_R$,*

$$\frac{\sqrt{1 + \mu^2 R^2} - 1}{\sqrt{1 + \mu^2 R^2} + 1} \leq \left| \frac{\sqrt{1 + \mu^2 N(X)} - 1}{\sqrt{1 + \mu^2 N(X)} + 1} \right| \leq \frac{1 - \sqrt{1 - \mu^2 R^2}}{\sqrt{1 - \mu^2 R^2} + 1}.$$

Define subspaces of \mathcal{H}_μ similar to \mathcal{H}^+ , \mathcal{H}^- and \mathcal{H}^0 :

$$\begin{aligned} \mathcal{H}_\mu^+ &= \mathbb{C} - \text{span of } \{f_{k,l,m,n}(Z); k \geq 0\}, \\ \mathcal{H}_\mu^- &= \mathbb{C} - \text{span of } \{f_{k,l,m,n}(Z); k \leq -(2l+2)\}, \\ \mathcal{H}_\mu^0 &= \mathbb{C} - \text{span of } \{f_{k,l,m,n}(Z); -(2l+1) \leq k \leq -1\} \end{aligned}$$

(and the ranges of indices l, m, n are $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, m, n = -l, -l+1, \dots, l$, as before). Thus $\mathcal{H}_\mu = \mathcal{H}_\mu^- \oplus \mathcal{H}_\mu^0 \oplus \mathcal{H}_\mu^+$.

Remark 45. *The spaces \mathcal{H}_μ , \mathcal{H}_μ^- , \mathcal{H}_μ^0 and \mathcal{H}_μ^+ are defined by analogy with spaces \mathcal{H}_μ and $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^0 \oplus \mathcal{H}^+$. We believe that these spaces can be characterized as images under the multiplication maps on $\mathcal{H}_\mu^\pm \otimes \mathcal{H}_\mu^\pm$. Thus \mathcal{H}_μ , \mathcal{H}_μ^- , \mathcal{H}_μ^0 and \mathcal{H}_μ^+ should be the images of $\mathcal{H}_\mu \otimes \mathcal{H}_\mu$, $\mathcal{H}_\mu^- \otimes \mathcal{H}_\mu^-$, $\mathcal{H}_\mu^- \otimes \mathcal{H}_\mu^+$ and $\mathcal{H}_\mu^+ \otimes \mathcal{H}_\mu^+$ respectively. (Compare with Lemma 8.)*

Note that

$$\mathcal{H}_\mu^- = \mathbb{C} - \text{span of } \{f'_{k,l,m,n}(Z); k \geq 0\}.$$

From our expansion of $(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^{-2}$ we immediately obtain the following analogue of Corollary 14:

Proposition 46. *Let $0 < R < \mu^{-1}$ and $r > 0$.*

1. *If $Y \in U(2)_r$ and*

$$\left| \frac{\sqrt{1 + \mu^2 N(Y)} - 1}{\sqrt{1 + \mu^2 N(Y)} + 1} \right| < \frac{\sqrt{1 + \mu^2 R^2} - 1}{\sqrt{1 + \mu^2 R^2} + 1},$$

then $r < R$ and the map

$$f \mapsto (P_\mu^+ f)(Y) = \frac{i}{2\pi^3} \int_{X \in U(2)_R} \frac{f(X) dV_{R,\mu}}{(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^2}, \quad f \in \mathcal{H}_\mu,$$

is a projector onto \mathcal{H}_μ^+ annihilating $\mathcal{H}_\mu^- \oplus \mathcal{H}_\mu^0$ and, in particular, provides a reproducing formula for functions in \mathcal{H}_μ^+ ;

2. *If $Y \in U(2)_r \cap \mathbb{U}_\mu$ and*

$$\frac{1 - \sqrt{1 - \mu^2 R^2}}{\sqrt{1 - \mu^2 R^2} + 1} < \left| \frac{\sqrt{1 + \mu^2 N(Y)} - 1}{\sqrt{1 + \mu^2 N(Y)} + 1} \right|,$$

then $r > R$ and the map

$$f \mapsto (P_\mu^- f)(Y) = \frac{i}{2\pi^3} \int_{X \in U(2)_R} \frac{f(X) dV_{R,\mu}}{(\langle \hat{X} - \hat{Y}, \hat{X} - \hat{Y} \rangle_{1,4})^2}, \quad f \in \mathcal{H}_\mu,$$

is a projector onto \mathcal{H}_μ^- annihilating $\mathcal{H}_\mu^0 \oplus \mathcal{H}_\mu^+$ and, in particular, provides a reproducing formula for functions in \mathcal{H}_μ^- .

The reproducing kernel and projector for the space \mathcal{H}_μ^0 can be obtained formally as in Section 5 and with full rigor as in Section 6. The advantage of the anti de Sitter deformation of \mathcal{H}^0 (and also \mathcal{H}^\pm) is that now we can extend the functions from this representation to the ambient five-dimensional Minkowski space $\mathbb{R}^{1,4}$, and we expect some additional flexibility in the permissible choices of integration cycles for the quaternionic analogues of Cauchy's formula for the second order pole (cf. Theorem 15 in this paper for the scalar case and Theorem 77 in [FL1] for the spinor case).

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